

Effective action from M-theory on twisted connected sum G_2 -manifolds

Thaïsa C. da C. Guio¹, Hans Jockers¹, Albrecht Klemm¹, Hung-Yu Yeh^{1,2}

¹ *Bethe Center for Theoretical Physics, Physikalisches Institut der Universität Bonn,
Nussallee 12, D-53115 Bonn, Germany*

² *Max Planck Institute for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany*

tguio@th.physik.uni-bonn.de

jockers@uni-bonn.de

aklemm@th.physik.uni-bonn.de

yeh@mpim-bonn.mpg.de

Abstract

We study the four-dimensional low energy effective $\mathcal{N} = 1$ supergravity theory of the dimensional reduction of M-theory on G_2 -manifolds, which are constructed by Kovalev's twisted connected sum gluing suitable pairs of asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ augmented with a circle S^1 . In the Kovalev limit the Ricci-flat G_2 -metrics are approximated by the Ricci-flat metrics on $X_{L/R}$ and we identify the universal modulus — the Kovalevton — that parametrizes this limit. We observe that the low energy effective theory exhibits in this limit gauge theory sectors with extended supersymmetry. We determine the universal (semi-classical) Kähler potential of the effective $\mathcal{N} = 1$ supergravity action as a function of the Kovalevton and the volume modulus of the G_2 -manifold. This Kähler potential fulfills the no-scale inequality such that no anti-de-Sitter vacua are admitted. We describe geometric degenerations in $X_{L/R}$, which lead to non-Abelian gauge symmetries enhancements with various matter content. Studying the resulting gauge theory branches, we argue that they lead to transitions compatible with the gluing construction and provide many new explicit examples of G_2 -manifolds.

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Glossary

Symbol	Description	Pages
Geometric spaces:		
$M^{1,10}$	eleven-dimensional Lorentz manifold (M-theory space-time)	5
$\mathbb{M}^{1,3}$	four-dimensional Minkowski space	5,7
Y	compact seven-dimensional compactification manifold	5,7,18,25
Y_{tw}	twisted connected sum G_2 -manifold from orthogonal gluing	38,40,45
\mathcal{M}	moduli space of Ricci-flat metrics of G_2 -metrics	9,14
$\mathcal{M}_{\mathbb{C}}$	semi-classical moduli space of M-theory on G_2 -manifolds	14
Δ^{cyl}	complex one-dimensional open cylinder $\{x \in \mathbb{C} \mid z > 1\}$	15
X^∞	cylindrical Calabi–Yau threefold	15
$X_{L/R}^\infty$	left/right cylindrical Calabi–Yau threefold	16
$X_{L/R}$	left/right asymptotically cylindrical Calabi–Yau threefold	16
$X_{L/R}(T)$	left/right truncated Calabi–Yau threefold $X_{L/R}$	18,17
$K_{L/R}$	left/right compact complement of asym. cyl. of $X_{L/R}(T)$	15,17
$S_{L/R}^{*1}$	left/right circle of the asymptotic region of $X_{L/R}$	18
$S_{L/R}^1$	left/right circles in Kovalev’s twisted connected sum	16,18
$Y_{L/R}^\infty$	left/right asymptotic seven-manifolds $X_{L/R}^\infty \times S_{L/R}^1$	16
$Y_{L/R}$	left/right non-compact seven-manifolds $X_{L/R} \times S_{L/R}^1$	16,25
$Y_{L/R}(T)$	left/right truncated non-compact seven-manifolds $Y_{L/R}$	18,17
S	K3 surface	15,21
$S_{L/R}$	left/right polarized K3 surface	16,18
(Z, S)	building block for asymptotically cylindrical Calabi–Yau threefold	23
(Z_{sing}, S)	singular building block with $\mathcal{N} = 2$ gauge theory sector	49,52
dP_ℓ	del Pezzo surface of degree $(9 - \ell)$	43
P	weak Fano, semi-Fano or Fano threefold	31,32
P_Σ	toric weak Fano, toric semi-Fano or toric Fano threefold	32,36
Σ	toric fan of a toric variety	32,36
Δ/Δ^*	toric and dual toric polytopes	32,36
Forms and tensors:		
$\eta_{\mu\nu}$	four-dimensional Minkowski metric	7
g_{mn}	Riemannian metric tensor of compactification manifold Y	7
\hat{g}	metric tensor of eleven-dimensional Lorentz manifold $M^{1,10}$	7
φ	G_2 -structure (three-form)	5,9
g_φ	G_2 -structure metric	6
$\omega_I^{(2)}$	basis of harmonic two-forms of G_2 -manifold Y	9,11
$\rho_i^{(3)}$	basis of harmonic three-forms of G_2 -manifold Y	8,9
$\rho_{i,(mn)}^{\text{sym}}$	basis of zero modes of the Lichnerowicz Laplacian on Y	8,11
η	covariantly constant spinor of G_2 -manifold Y	10
ω^∞	Kähler form of cylindrical Calabi–Yau threefold X^∞	15
Ω^∞	holomorphic three-form of cylindrical Calabi–Yau threefold X^∞	15
g_{X^∞}	Ricci-flat metric of cylindrical Calabi–Yau threefold X^∞	15
$g_{L/R}$	left/right Ricci-flat metric of Calabi–Yau threefold $X_{L/R}$	20

Symbol	Description	Pages
Forms and tensors (<i>continued</i>):		
$\omega_{L/R}^{I,J,K}$	left/right triplet of hyper Kähler two-forms of K3 surface $S_{L/R}$	16
g_S	Ricci-flat metric of K3 surface S	15
φ_0	canonical G_2 -structure of Calabi–Yau threefold times circle	16
$\varphi_{0L/R}^\infty$	left/right asymptotic torsion-free G_2 -structure	17
$\tilde{\varphi}_{L/R}(\gamma, T)$	left/right interpolating G_2 -structure	19
$\varphi(\gamma, T)$	torsion-free G_2 -structure in Kovalev’s twisted connected sum	19
Coordinates and quantum fields:		
x^μ	coordinates of Minkowski space $\mathbb{M}^{1,3}$	7
y^m	local coordinates of compact seven-manifold Y	7
S^i	local coordinates of moduli space \mathcal{M}	9,11
P^i	three-form scalar fields	9,11
ϕ^i	local coordinates of moduli space \mathcal{M}_C & chiral scalar fields	13
χ_α^i	four-dimensional chiral fermions of $\mathcal{N} = 1$ chiral multiplets	11
Φ^i	four-dimensional $\mathcal{N} = 1$ chiral multiplets	11,13
A_μ^I	four-dimensional vector bosons	9,11
λ_α^I	four-dimensional gauginos	11
V^I	four-dimensional $\mathcal{N} = 1$ vector multiplets	11
ν	four-dimensional $\mathcal{N} = 1$ chiral overall volume modulus	26,29
\varkappa	Kovalevton (four-dimensional $\mathcal{N} = 1$ chiral gluing modulus)	26,29
Parameters and coupling constants:		
κ_4/κ_{11}	four-dimensional/eleven-dimensional Planck constant	13
V_{Y_0}	constant reference volume of manifold Y	12,21
V_Y	moduli-dependent volume of manifold Y	12
λ_0	moduli-dependent dimensionless volume factor of manifold Y	12
γ_0	constant reference radius	20
γ	moduli-dependent radius	16,20
λ	inverse length scale of Calabi–Yau threefolds $X_{L/R}^\infty$	16,21
$\tilde{\lambda}$	dimensionless inverse length scale of Calabi–Yau threefolds $X_{L/R}^\infty$	21
λ_S	inverse length scale of K3 surface S	16
R	dimensionless volume modulus	20,22
T	dimensionless Kovalev parameter	18,17,22
\tilde{S}	dimensionless non-universal moduli	21,26
$t_{L/R}$	left/right Kähler moduli of Calabi–Yau threefold $X_{L/R}$	20
$z_{L/R}$	left/right comp. struct. moduli of Calabi–Yau threefold $X_{L/R}$	20
Cohomology groups and lattices:		
L	two-form lattice of K3 surfaces $S_{L/R}$, $L = H^2(S_L, \mathbb{Z}) = H^2(S_R, \mathbb{Z})$	24
$N_{L/R}$	left/right Picard lattices of polarized K3 surface $S_{L/R}$	24
$T_{L/R}$	left/right transcendental lattices of polarized K3 surface $S_{L/R}$	24
$k_{L/R}$	left/right kernels of two-form cohomology	24
W	orthogonal pushout lattice	33
$W_{L/R}$	left/right orthogonal complement to the intersection lattice R	34
R	intersection lattice	33

Symbol	Description	Pages
Cohomology groups and lattices (<i>continued</i>):		
$N_L \perp_R N_R$	orthogonal pushout of lattices $N_{L/R}$ at the intersection lattice R	33
$N_L \perp N_R$	perpendicular gluing of lattices $N_{L/R}$	35
$\langle \cdot, \cdot \rangle$	lattice intersection pairing	33
κ	lattice intersection matrix	37,40,42
Δ^κ	discriminant of the lattice intersection matrix	37,40,42
Gauge theory data:		
$(\cdot)^\flat$	superscript for Higgs branch quantity	50,54
$(\cdot)^\sharp$	superscript for Coulomb branch quantity	50,54
H^\flat	$\mathcal{N} = 2$ Higgs branch	50,54
C^\sharp	$\mathcal{N} = 2$ Coulomb branch	50,54
h^\flat	complex dimension of the Higgs branch	50,54
c^\sharp	complex dimension of the Coulomb branch	50,54
G	gauge group	49,53
adj	adjoint representation	53,57,58
k	fundamental representation of $SU(k)$	53,57,58
Miscellaneous:		
Δ	Laplacian of compact seven-manifold Y	8,10
Δ_L	Lichnerowicz Laplacian of compact seven-manifold Y	8,10
\not{D}	Dirac operator of compact seven-manifold Y	10,70
\not{D}^{RS}	Rarita–Schwinger operator of compact seven-manifold Y	10,70
K	Kähler potential of $\mathcal{N} = 1$ supergravity action	13,29,30
f_{IJ}	gauge kinetic coupling functions of $\mathcal{N} = 1$ supergravity action	13
W	superpotential of $\mathcal{N} = 1$ supergravity action	14
$(\cdot)_{L/R}$	left/right subscript in Kovalev’s twisted connected sum	16,18
F_Λ	gluing diffeomorphism in Kovalev’s twisted connected sum	17
r	hyper Kähler rotation mapping polarized K3 surface S_L to S_R	17
$\text{MM}\#_\rho$	reference to rank ρ Fano threefold in the Mori–Mukai classification	33
$\text{K}\#$	reference to toric semi-Fano threefolds in the Kasprzyk list	33

1 Introduction

M-theory compactifications on seven-dimensional manifolds with G_2 holonomy offer the opportunity to geometrically study the properties of $\mathcal{N} = 1$ effective theories in a setting that is non-perturbative from the superstring point of view [1–4]. As M-theory is conjectured to be the non-perturbative extension of type IIA theory, it is natural to compare it to F-theory compactifications on elliptically-fibered Calabi–Yau fourfolds. It leads to effective $\mathcal{N} = 1$ theories in four dimensions and is the geometrization of non-perturbative type IIB compactifications on the complex three dimensional base of the fibration. It also includes a varying axio-dilaton background due to space-time-filling seven branes.

While standard techniques of complex algebraic geometry provide immediately hundreds of thousands of elliptically-fibered Calabi–Yau fourfolds — for instance realized as hypersurfaces and complete intersections in weighted projective spaces or more generally in toric ambient spaces [5, 6] — for a long time there were only about a hundred examples of G_2 -manifolds constructed by the resolution of special orbifolds of seven-dimensional torus T^7 [7]. Likewise, the holomorphic terms in the four-dimensional low energy effective $\mathcal{N} = 1$ supergravity action obtained from Calabi–Yau fourfolds are computable in the underlying algebraic setting. In particular, the flux-induced superpotential is essentially determined by the integral periods of the Calabi–Yau fourfolds [8], which for compact fourfolds have systematically been determined in refs. [9, 10]. Furthermore, the holomorphic gauge kinetic coupling functions are in principle accessible in this setting as well as threshold corrections to the gauge kinetic terms [11, 12].¹

M-theory, however, has the clear advantage that — at least in the supergravity limit — we expect an explicit description in terms of eleven-dimensional supergravity, for which a unique eleven-dimensional supergravity action exists [14].² Therefore, by simply studying Kaluza–Klein reductions of this action on compact seven-dimensional manifolds one obtains four-dimensional low-energy effective theories that capture already many of the physical properties of the associated M-theory compactifications [1, 2, 16–20]. The resulting semi-classical four-dimensional effective action is then further corrected by non-perturbative effects specific to M-theory, such as M2- and M5-brane instantons wrapping internal cycles of the seven-dimensional compactification manifold [21, 22].

More recently a new construction of G_2 -manifolds has been proposed by Kovalev [23], which we loosely refer to as Kovalev’s twisted connected sum. The essential idea is to consider suitable pairs of non-compact asymptotically cylindrical

¹The gauge kinetic coupling functions are holomorphic and one loop exact. However, physical gauge kinetic terms receive further corrections in $\mathcal{N} = 1$ effective theories, see for instance ref. [13] for a thorough discussion.

²The maximal dimension that allows this supersymmetry representation was previously pointed out in ref. [15].

Calabi–Yau threefolds times a circle — referred to as building blocks — that are glued in their asymptotic region to compact seven-dimensional manifolds admitting a Ricci-flat metric of G_2 holonomy. Further explorations of this construction show that Kovalev’s twisted connected sum offers again a large number of explicit examples of G_2 -manifolds [24–26]. In fact the origin of this multitude is similarly based on toric geometry as the asymptotically cylindrical Calabi–Yau threefolds — which furnish the distinct summands in Kovalev’s twisted connected sum — can for instance be constructed using the toric weak Fano threefolds, defined by the 4 319 reflexive polytopes in three dimensions, by blowing up a suitable curve and removing the anti-canonical class [27]. Currently there is no systematic gluing prescription for the whole class available, but admissible gluing conditions can be straightforwardly established for building blocks constructed from 899 toric varieties of the semi-Fano type defined in [24]. They admit in general different Kähler cones and — as discussed further in this work — they can be degenerated and resolved to yield different building blocks realizing distinct branches of both Abelian and non-Abelian gauge theories. An estimate — even accounting for the possibility that the homeomorphism class of the constructed examples occur multiple times — yields nevertheless a factor of ten for each admissible building block. This leads to an estimated number of $10^8 \times m_g$ different G_2 examples, where $m_g \geq 1$ is multiplicity due to the different gluings. A rough estimate for non-homeomorphic elliptically-fibered Calabi–Yau fourfolds yields alone 10^{18} hypersurfaces in toric varieties.³ The still huge difference in the orders of magnitude is maybe due to the fact that Kovalev’s twisted connected sum only realizes a particular class of G_2 -manifolds. Namely, Crowley and Nordström define a (non-trivial) \mathbb{Z}_{48} -valued homotopy invariant for G_2 -manifolds that takes the value 24 for any twisted connected sum G_2 -manifold [29].

One goal of this paper is to study the four-dimensional $\mathcal{N} = 1$ low energy effective action that arises from M-theory compactifications on G_2 -manifolds that are of the twisted connected sum type. In order to determine the defining data of the resulting $\mathcal{N} = 1$ supergravity theory — such as the Kähler potential, the gauge kinetic coupling functions, and the superpotential — an important question is to which extent the harmonic analysis of the asymptotically cylindrical Calabi–Yau threefold summands with their Ricci-flat Calabi–Yau metrics approximates the one of the Ricci-flat G_2 -metric of the resulting compact G_2 -manifold. In a certain limit — to be referred to as the Kovalev limit in the following — the corrections to the G_2 -metric expressed in terms of the Calabi–Yau data become exponentially suppressed [23, 24, 30, 27]. Thus, it is the Kovalev limit that allows us to reliably deduce from the geometry of the Calabi–Yau summands the resulting low-energy effective action. To a some extent we can think of the Kovalev limit of M-theory on G_2 -manifolds as the analog of the large

³This is based on the observation that the number of d -dimensional reflexive polytopes grows at least exponentially in d . Mark Gross’s proof for the finiteness of elliptically-fibered Calabi–Yau threefolds in ref. [28] suggests that the estimated number of elliptically-fibered Calabi–Yau fourfolds is again finite and the order of magnitude independent of the construction.

volume limit of Calabi–Yau compactifications of type II string theories.

The topological analysis of the Kovalev’s twisted connected sum gives a rather detailed description on how the cohomology of the resulting G_2 -manifold is constructed from the (relative) cohomology of the asymptotically cylindrical Calabi–Yau threefold summands [23, 24]. This cohomological data determines the $\mathcal{N} = 1$ vector and chiral multiplets of the resulting four-dimensional theory. In the Kovalev limit we find that the $\mathcal{N} = 1$ vector multiplets furnish gauge theory sectors of extended supersymmetry. Specifically, $\mathcal{N} = 1$ vector multiplets attributed to the interior of the asymptotically Calabi–Yau threefolds combine with $\mathcal{N} = 1$ chiral multiplets to $\mathcal{N} = 2$ vector multiplets realizing $\mathcal{N} = 2$ gauge theory sector, whereas $\mathcal{N} = 1$ vector multiplets associated to the mutual asymptotic region of the Calabi–Yau summands enhance to $\mathcal{N} = 4$ gauge theory sectors.

In the spectrum of twisted connected sum G_2 -manifolds we identify two universal $\mathcal{N} = 1$ chiral fields ν and \varkappa . The real part of the chiral field ν furnishes the overall volume modulus of the G_2 -manifold. The chiral field \varkappa is specific to Kovalev’s twisted connected sum construction, as its real part parametrizes the Kovalev limit. In the sequel we refer to this multiplet as the Kovalevton \varkappa . Restricting the dynamics of the two universal chiral moduli fields in the vicinity of the Kovalev limit, we arrive for the Kähler potential of the four-dimensional effective supergravity theory at the universal expression

$$K = -\log [(\nu + \bar{\nu})^4 (\varkappa + \bar{\varkappa})^3] . \quad (1.1)$$

This simple semi-classical Kähler potential — only capturing the dynamics of the two universal chiral multiplets — fulfills the no-scale inequality implying a manifest non-negative F-term scalar potential, such that no (supersymmetric) anti-de-Sitter vacua can occur.

From a physics point of view the most interesting question is whether the twisted sum construction can accommodate singularities that in the four-dimensional effective theory lead to enhanced non-Abelian gauge symmetries, to geometrically engineerable matter content, to a non-chiral spectrum and to transitions within the class $\mathcal{N} = 1$ effective theories connecting topologically inequivalent G_2 -manifolds. The first three questions have been addressed in refs. [31–35] and the last one in refs. [36, 37, 25], however, mainly in the context of local models. The Kovalev construction and the above described decoupling into sectors with different amount of supersymmetry allow us to discuss some of these questions in particular the first two and the last one in the context of global G_2 -manifolds. More explicitly, we find that by degenerating the certain algebraic equations in the description of the building blocks and blowing up the corresponding singularities that we can achieve various Abelian and non-Abelian gauge symmetries — for example including the standard model gauge group — and matter in the adjoint, bi-fundamental and fundamental representations. Moreover, following refs. [38–43] we analyze the Higgs and Coulomb branches of the $\mathcal{N} = 2$ gauge theory sectors to realize transitions in the building blocks of twisted connected sum G_2 -

manifolds. The remarkable fact is now that the predicted spectra in the various gauge theory branches agree with the changes among the $\mathcal{N} = 1$ supergravity spectra of the corresponding compact G_2 -manifolds. This leads us to propose geometric transitions among G_2 -manifolds that are physically connected via branches of $\mathcal{N} = 2$ gauge theory sectors.

The paper is organized as follows: In Section 2 we review the geometry of the G_2 -manifolds and the Kaluza–Klein reduction of eleven-dimensional supergravity on these spaces. We focus on the moduli space of such G_2 -compactifications, on the four-dimensional low energy effective spectrum and action, and on the resulting four-dimensional $\mathcal{N} = 1$ supergravity description in terms of the Kähler potential, the (flux-induced) superpotential, and the gauge kinetic coupling functions. Section 3 reviews Kovalev’s twisted connected sum construction. Firstly, we introduce the asymptotically cylindrical Calabi–Yau threefolds, and secondly we summarize the twisted connected sum constructed from a suitable pair of such Calabi–Yau threefolds. Due to the importance to our analysis, a particular emphasis is put on the Kovalev limit. In Section 4 we describe M-theory compactifications on twisted connected sum G_2 -manifolds. We start with a description of the low-energy effective $\mathcal{N} = 1$ spectrum as deduced from the cohomology of the Calabi–Yau summands. We analyze the universal properties of the low-energy effective theory attributed to Kovalev’s twisted connected sum. In Section 5 we apply the method of orthogonal gluing to explicitly construct novel examples of twisted connected G_2 -manifolds. We argue that in the Kovalev limit these examples directly relate to Abelian $\mathcal{N} = 4$ gauge theory sectors. In Section 6 we study the emergence of Abelian and non-Abelian $\mathcal{N} = 2$ gauge theory sectors. For both the Abelian and non-Abelian $\mathcal{N} = 2$ gauge theory sectors we establish a correspondence between $\mathcal{N} = 2$ Higgs and Coulomb branches of the gauge theory and the associated phases of twisted connected G_2 -manifolds. We illustrate the different physical aspects of the proposed correspondence with explicit examples of G_2 -manifolds. Our conclusions are presented in Section 7. For the convenience of the reader, we collect in the glossary on page ii to iv our notational conventions of the used mathematical symbols together with a reference to their appearance in the main text. In Appendix A we give further technical details on the G_2 compactifications of fermionic terms, supplementing the material presented in Section 2.

2 M-theory on G_2 -manifolds

An eleven-dimensional Lorentz manifold $M^{1,10}$ together with a four-form flux G of an anti-symmetric three-form tensor field \hat{C} describes the geometry of the low energy effective action of M-theory. Firstly — due to fermionic degrees of freedom in M-theory — the Lorentz manifold $M^{1,10}$ must be spin. Secondly — consistency of the effective

action at one loop — imposes the cohomological flux quantization condition [44]

$$\frac{G}{2\pi} - \frac{\lambda}{2} \in H^4(M^{1,10}, \mathbb{Z}) , \quad \lambda = \frac{p_1(M^{1,10})}{2} . \quad (2.1)$$

The class λ is integral since the first Pontryagin class p_1 is even for seven-dimensional spin manifolds.

In this work we study compactifications of M-theory to four-dimensional Minkowski space $\mathbb{M}^{1,3}$ with $\mathcal{N} = 1$ supersymmetry. That is to say that for the eleven-dimensional Lorentz manifold $M^{1,10}$ we consider the compactification ansatz

$$M^{1,10} = \mathbb{M}^{1,3} \times Y \quad (2.2)$$

with the seven-dimensional compact smooth manifold Y . In the absence of background fluxes such a four-dimensional $\mathcal{N} = 1$ Minkowski vacuum implies that the internal space Y must be a G_2 manifold [1, 3, 4].

A G_2 -manifold Y is a seven-dimensional Ricci-flat Riemannian manifold with G_2 holonomy and not a proper subgroup thereof. Furthermore, the manifold Y is spin with a single globally defined covariantly constant spinor [45]. Note that for G_2 -manifolds the characteristic class λ is always even [22, 17], which is consistent with the quantization condition (2.1) for the considered compactification ansatz of vanishing background flux.

Let us briefly recall some relevant aspects of G_2 -manifolds. First of all, the exceptional Lie group G_2 is a fourteen-dimensional simply connected subgroup of $SO(7)$, and we can think of G_2 in the following way. A three-form φ on \mathbb{R}^7 gives rise to a canonical symmetric bilinear form on \mathbb{R}^7

$$B_\varphi(X, Y) = -\frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \quad (2.3)$$

with values in $\Lambda^7(\mathbb{R}^7)^*$. For a generic three-form φ , the bilinear form B_φ yields a non-degenerate pairing of some signature (p, q) with $p + q = 7$ (with respect to an oriented volume form on \mathbb{R}^7) [46, 47]. In particular, there is an open set $\Lambda_+^3(\mathbb{R}^7)^*$ in the space of three-forms $\Lambda^3(\mathbb{R}^7)^*$ such that B_φ is a positive definite bilinear form for $\varphi \in \Lambda_+^3(\mathbb{R}^7)^*$. Then $GL(7, \mathbb{R})$ acts on the three-form φ and G_2 is its fourteen-dimensional stabilizer subgroup. Since G_2 leaves the positive definite pairing B_φ invariant, it is actually a subgroup of $SO(7)$.

Since the Lie group G_2 is the stabilizer group of the described three-form, a seven-dimensional oriented manifold Y together with the three-form φ in $\Omega_+^3(Y)$ — which is the space of smooth three-forms oriented-isomorphic to $\Lambda^3 T_p^* Y \simeq \Lambda_+^3(\mathbb{R}^7)^*$ for any $p \in M$ — becomes a G_2 -structure manifold.⁴ Thus we call φ a G_2 -structure on Y .

⁴Note that $\Lambda_+^3(\mathbb{R}^7)^*$ is a convex open set in $\Lambda^3(\mathbb{R}^7)^*$. Thus — with a partition of unity — we can construct a G_2 -structure on any smooth paracompact seven-dimensional manifold Y .

Furthermore, the positive definite pairing (2.3) defines a Riemannian metric g_φ on Y . Namely, at any point $p \in M$ and for any basis $\partial_1|_p, \dots, \partial_7|_p$ at $T_p Y$, we obtain the positive definite inner product

$$\begin{aligned} g_\varphi(X_p, Y_p) &= \frac{B_\varphi(X_p, Y_p)(\partial_1|_p, \dots, \partial_7|_p)}{\text{vol}_\varphi(\partial_1|_p, \dots, \partial_7|_p)} , \\ \text{vol}_\varphi(\partial_1|_p, \dots, \partial_7|_p)^9 &= \det [B_\varphi(\partial_i|_p, \partial_j|_p)(\partial_1|_p, \dots, \partial_7|_p)] , \end{aligned} \quad (2.4)$$

for vectors X_p and Y_p in the tangent space $T_p Y$.

The remarkable and important theorem for the following is that a G_2 -structure manifold has a subgroup of G_2 as its holonomy group if and only if the three-form φ is harmonic with respect to the constructed G_2 metric g_φ [45], i.e.,

$$d\varphi = 0 , \quad d *_{g_\varphi} \varphi = 0 , \quad (2.5)$$

in terms of the Hodge star $*_{g_\varphi}$ of the metric g_φ . Such a harmonic three-form in $\Omega_+^3(Y)$ is called torsion-free. Requiring in addition a finite fundamental group $\pi_1(Y)$ ensures that Y has G_2 holonomy and not a proper subgroup thereof. Thus these manifolds — that is to say with finite fundamental group $\pi_1(Y)$ and torsion-free G_2 -structure — are referred to as G_2 -manifolds.

The system of partial differential equations (2.5) for torsion-free G_2 -structures are highly non-linear due to relation (2.4) between the metric g_φ and the G_2 -structure φ . Nevertheless, given a torsion-free G_2 -structure of a G_2 -manifold, the local structure of the moduli space \mathcal{M} of G_2 -manifolds is known due to Joyce [7]. In particular the Betti number $b_3(Y)$ is the dimension of \mathcal{M} . We will discuss these aspects of the local structure in the next section.

2.1 Kaluza–Klein reduction on G_2 -manifolds

Eleven-dimensional $\mathcal{N} = 1$ supergravity compactified on a seven-dimensional manifold Y without four-form background fluxes to four-dimensional supergravity has been first discussed in ref. [1]. A possible warp factor in this compactification has been considered in ref. [2], where it was shown that the wrapping breaks supersymmetry.⁵ The structure of the massless four-dimensional $\mathcal{N} = 1$ multiplets that arise from such compactification was shown to possess $b_2(Y)$ abelian $U(1)$ vector fields and $b_3(Y)$ neutral chiral fields Φ^i in ref. [4], see also ref. [3]. The inclusion of background fluxes G — generating a superpotential W for the neutral chiral fields Φ^i and thereby generically breaking supersymmetry — has been analyzed in refs. [16, 17].

⁵While these authors give the criteria for the compactification manifold Y to yield four-dimensional $\mathcal{N} = 1$ supergravity, they do not refer to the G_2 -manifolds in Berger’s classification of special holonomy manifolds [48], likely because compact examples of G_2 -manifolds were only found much later in ref. [7].

Let us now review the Kaluza–Klein reduction of eleven-dimensional $\mathcal{N} = 1$ supergravity, which furnishes the low-energy effective description of M-theory. The massless spectrum of this maximally supersymmetric supergravity theory consists only of the eleven-dimensional gravity multiplet. Its bosonic massless field content is given by the eleven-dimensional space-time metric tensor \hat{g}_{MN} , the three-form tensor $\hat{C}_{[MNP]}$, whereas the fermionic massless field content is given by the eleven-dimensional gravitino $\hat{\Psi}_M^\alpha$. The degrees of freedom of the massless component fields in the gravity multiplet transform in the following irreducible representations of the little group $SO(9)$:

- The metric \hat{g}_{MN} in the traceless symmetric representation **44**.
- The three-form $\hat{C}_{[MNP]}$ in the anti-symmetric three-tensor representation **84**.
- The gravitino $\hat{\Psi}_M^\alpha$ in the spinorial representation **128_s**.

We now perform the Kaluza–Klein reduction to four-dimensional Minkowski space $\mathbb{M}^{1,3}$ with the compactification ansatz (2.2). To solve Einstein equations in the absence of background fluxes, we consider the block diagonal metric⁶

$$\hat{g}(x, y) = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n, \quad (2.6)$$

where x^μ and y^m furnish (local) coordinates of the four-dimensional Minkowski space $\mathbb{M}^{1,3}$ with the flat space-time metric $\eta_{\mu\nu}$ and the seven-dimensional G_2 -manifold Y with the Ricci-flat Riemannian metric g_{mn} , respectively. Notice that we use upper-case latin letters for 11-dimensional indices, lower-case latin letters for 7-dimensional indices, and greek letters for 4-dimensional indices.

The first task is to deduce the massless spectrum of the effective four-dimensional low energy theory. We start with the gravitational degrees of freedom, which infinitesimally describe the fluctuations of the metric background (2.6), i.e., $\hat{g} \rightarrow \hat{g} + \delta\hat{g}$. Firstly, we obtain the four-dimensional metric fluctuations $\delta g_{\mu\nu}$, which corresponds to the gravitational degrees of the four-dimensional low-energy effective theory. Secondly, since the fundamental group of G_2 -manifolds is finite, there are no massless gravitational Kaluza–Klein vectors. Finally, we determine the gravitational Kaluza–Klein scalars S^i , which furnish coordinates on the moduli space of G_2 -metrics. At a given point S^i in the moduli space we fix a reference metric, and consider its infinitesimal deformation under δS^i , i.e.,

$$g_{mn}(S^i) dy^m dy^n \rightarrow g_{mn}(S^i) dy^m dy^n + \sum_i \delta S^i \rho_{i,(mn)}^{\text{sym}}(S^i) dy^m dy^n. \quad (2.7)$$

⁶In the presence of background fluxes in the internal space Y , the ansatz for the metric is generalized to a warped product $\hat{g}(x, y) = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} g_{mn} dy^m dy^n$ in terms of the function $A(y)$ on Y called the warped factor [16], which generically breaks four-dimensional $\mathcal{N} = 1$ supersymmetry [2, 16].

Then solving Einstein equations to linear order in the *symmetric metric fluctuations* $\rho_{i,(mn)}^{\text{sym}}$, we obtain

$$\text{Ric} \left(g + \sum \delta S^i \rho_i^{\text{sym}} \right) = 0 \quad \Rightarrow \quad \Delta_L \rho_i^{\text{sym}} = 0 , \quad (2.8)$$

in terms of the Lichnerowicz Laplacian Δ_L for the symmetric tensor fields. Using the G_2 -structure φ on Y , we construct the anti-symmetric three-form tensors

$$\rho_{i,[mnp]}^{(3)} = g^{rs} \rho_{i,r[m}^{\text{sym}} \varphi_{np]s} . \quad (2.9)$$

On G_2 -manifolds the symmetric tensor ρ_i^{sym} is a zero mode to the Lichnerowicz Laplacian operator if and only if the above constructed three-form $\rho_i^{(3)}$ is harmonic [49], namely

$$\Delta_L \rho_i^{\text{sym}} = 0 \quad \Leftrightarrow \quad \Delta \rho_i^{(3)} = 0 . \quad (2.10)$$

Thus, the massless Kaluza–Klein scalars S^i arise from harmonic three-forms $\rho_i^{(3)}$, which represent a basis for the vector space $H^3(Y)$ of dimension $b_3(Y)$. According to eq. (2.5) the harmonic three-forms $\rho_i^{(3)}$ are the first order deformations to the torsion free G_2 -structure

$$\varphi(S^i) \rightarrow \varphi(S^i) + \sum_i \delta S^i \rho_i^{(3)}(S) . \quad (2.11)$$

At a given point S^i in moduli space the harmonic three-forms $\rho_i^{(3)}$ of Y fall into representations of the structure group G_2 , and $H^3(Y)$ splits as [7]

$$H^3(Y) = H_1^3(Y) \oplus H_{27}^3(Y) , \quad \dim H_1^3(Y) = 1 , \quad \dim H_{27}^3(y) = b_3(Y) - 1 , \quad (2.12)$$

where the three-form representatives transform in the representations **1** and **27** of G_2 , respectively. The harmonic torsion-free G_2 -structure φ corresponds to the unique singlet, and the associated deformation simply rescales the volume of the G_2 -manifold Y . The remaining harmonic forms in the representation **27** infinitesimally deform the torsion-free G_2 -structure such that the volume of Y remains constant at first order approximation. Analogously, the symmetric tensors ρ_i^{sym} solving the Lichnerowicz Laplacian Δ_L split into a unique singlet — given by the metric tensors g — and $b_3(Y) - 1$ traceless symmetric tensors in the representation **27** of the G_2 -structure group.

Above we have seen that the infinitesimal deformations can be identified with harmonic three-forms. In ref. [7] Joyce shows that these infinitesimal deformations are actually unobstructed to all orders. That is to say that the vicinity $U_{\varphi(S^i)} \subset \mathcal{M}$ of a given torsion-free G_2 -structure $\varphi(S^i) \in \mathcal{M}$ — at a given point S^i , $i = 1, \dots, b_3(Y)$, in the moduli space — is locally diffeomorphic to the de Rham cohomology $H^3(Y)$,⁷

⁷We assume that the scalars S^i describe a generic point in \mathcal{M} . First of all, the associated G_2 -manifold Y should be smooth. Furthermore, it should not be a special symmetric point corresponding to an orbifold singularity in \mathcal{M} .

i.e.,

$$\mathcal{P}_{\varphi(S^i)} : U_{\varphi(S^i)} \subset \mathcal{M} \rightarrow H^3(Y) , \quad \varphi \mapsto [\varphi] . \quad (2.13)$$

Hence, the Betti number $b^3(Y)$ is indeed the dimension of \mathcal{M} , and the scalar field S^i furnish local coordinates on \mathcal{M} with the infinitesimal deformations δS^i spanning the tangent space $T_{S^i}\mathcal{M}$. This local structure implies that the massless infinitesimally metric deformations ρ_i^{sym} — or alternatively the first order deformations $\rho_i^{(3)}$ to the torsion-free G_2 -structure — extend order-by-order to unobstructed finite deformations, which therefore describe locally the moduli space \mathcal{M} of G_2 -manifolds. While the harmonic three-forms $\rho_i^{(3)}$ themselves depend (non-linearly) on the moduli space coordinates S^i , due to eq. (2.13) we can locally expand the cohomology class $[\varphi]$ of the torsion-free G_2 -structure φ as

$$[\varphi(S^i)] = \sum_i S^i [\rho_i^{(3)}] , \quad (2.14)$$

which is a useful local description of the moduli space of Y .⁸

Massless four-dimensional modes arise from the coefficients in the decomposition of the eleven-dimensional anti-symmetric three-form tensor \hat{C} as

$$\hat{C}(x, y) = \sum_I A^I(x) \wedge \omega_I^{(2)}(y) + \sum_i P^i(x) \wedge \rho_i^{(3)}(y) , \quad (2.15)$$

in terms of the harmonic two-forms $\omega_I^{(2)}$ and three-forms $\rho_i^{(3)}$ identified with non-trivial cohomology representatives of $H^2(Y)$ and $H^3(Y)$ of dimension $b_2(Y)$ and $b_3(Y)$, respectively. Thus, as there are no dynamical degrees of freedom in four-dimensional anti-symmetric three-form tensor fields and due to the absence of harmonic one-forms on the internal G_2 -manifolds, the four-dimensional vectors A^I , $I = 1, \dots, b_2(Y)$, and the four-dimensional scalars P^i , $i = 1, \dots, b_3(Y)$, are the only massless modes obtained from the dimensional reduction of the eleven-dimensional anti-symmetric three-form tensor field \hat{C} .

Let us now turn to the dimensional reduction of the eleven-dimensional gravitino $\hat{\Psi}$, which geometrically is a section of $T^*M^{1,10} \otimes SM^{1,10}$, where $SM^{1,10}$ denotes a spin bundle of $M^{1,10}$. Upon dimensional reduction the gravitino $\hat{\Psi}$ enjoys the expansion

$$\hat{\Psi}(x, y) = (\psi_\mu(x) dx^\mu + \psi_\mu^*(x) dx^\mu) \zeta(y) + (\chi(x) + \chi^*(x)) \zeta_n^{(1)}(y) dy^n . \quad (2.16)$$

Here (ψ_μ, ψ_μ^*) and (χ, χ^*) are four-dimensional Rarita–Schwinger and four-dimensional spinor fields of both chiralities in $\mathbb{M}^{1,3}$.⁹ ζ is a section of the (real) spin bundle SY

⁸Note, however, that a given cohomology class $[\varphi(S^i)]$ is not necessarily represented by a unique torsion-free G_2 -structure $\varphi(S^i)$. Mathematically, not much is known about the global structure of the moduli space \mathcal{M} and, in particular, about the global map $\mathcal{P} : \mathcal{M} \rightarrow H^3(Y)$.

⁹In our conventions the fermionic fields ψ_μ , χ and ψ_μ^* , χ^* are chiral and anti-chiral, respectively, such that $\psi_\mu + \psi_\mu^*$ and $\chi + \chi^*$ become Majorana fermions.

of the compact G_2 -manifold Y .¹⁰ Furthermore, $\zeta^{(1)}$ is a section of the (real) Rarita–Schwinger bundle $T^*Y \otimes SY$, which locally takes the form $\theta^{(1)} \otimes \tilde{\zeta}$ in terms of the local one-form $\theta^{(1)}$ and the spinorial section $\tilde{\zeta}$.

On the G_2 -manifold the spin bundle splits as $SY \simeq T^*Y \oplus \mathbb{R}$ [7], such that the section ζ decomposes accordingly

$$\zeta = \sum_m a_m(y) \gamma^m \eta + b(y) \eta . \quad (2.17)$$

Here, η is the covariantly constant Majorana spinor of the G_2 -manifold and γ^m are the seven-dimensional gamma matrices. Similarly, we analyze the Rarita–Schwinger section $\zeta^{(1)}$ of $T^*Y \otimes SY$. It decomposes as

$$\zeta^{(1)} = \sum_{n,m} a_{(nm)}^{28}(y) dy^n \otimes \gamma^m \eta + \sum_{n,m} a_{[nm]}^{14}(y) dy^n \otimes \gamma^m \eta + \sum_n b_n^7(y) dy^n \otimes \eta . \quad (2.18)$$

The superscripts in the symmetric tensor $a_{(nm)}^{28}(y)$, the anti-symmetric tensor $a_{[nm]}^{14}(y)$, and the vector $b_n^7(y)$ indicate the dimension of their respective representations with respect to the structure group G_2 . While the anti-symmetric tensor $a_{[nm]}^{14}(y)$ and the vector $b_n^7(y)$ transform in the irreducible representations **14** and **7**, the symmetric tensor $a_{(nm)}^{28}(y)$ further decomposes into the trace and the traceless symmetric part, which respectively correspond to the irreducible representations **1** and **27**.

The massless four-dimensional fermionic spectrum results from the zero modes of the seven-dimensional Dirac operator \not{D} and Rarita–Schwinger operator \not{D}^{RS} , i.e.,

$$\not{D}\zeta = 0 , \quad \not{D}^{\text{RS}}\zeta^{(1)} = 0 . \quad (2.19)$$

The zero modes of these operators on G_2 -manifolds are discussed in Appendix A.3 and are also determined in ref. [50]. For the spinorial section ζ , the covariantly constant spinor η — i.e., $b(y) \equiv 1$ — is the only zero mode of the Dirac operator. In the Rarita–Schwinger section, $\zeta^{(1)}$ the one-form tensor $b^7(y) = b_n^7(y) dy^n$ does not contribute any zero modes. All zero modes arise from the zero modes of the Lichnerowicz Laplacian and the two-form Laplacian acting respectively on the symmetric tensors $a^{28}(y) = a_{(nm)}^{28}(y) dy^n \otimes dy^m$ and the anti-symmetric tensors $a^{14}(y) = a_{[nm]}^{14}(y) dy^n \wedge dy^m$, i.e.,

$$\Delta_L a^{28}(y) = 0 , \quad \Delta a^{14}(y) = 0 . \quad (2.20)$$

The zero modes of the Lichnerowicz Laplacian on G_2 -manifolds are again identified with harmonic three-forms according to eqs. (2.9) and (2.10) — with a single zero mode and $b_3(Y) - 1$ traceless symmetric zero modes transforming in the G_2 -representations **1** and **27**, respectively. Therefore, the zero modes of the Rarita–Schwinger bundle

¹⁰As the G_2 -structure group of Y — a subgroup of $SO(7)$ — is simply connected, it defines a canonical spin structure on Y .

Multiplicity	Massless 4d component fields		Massless 4d $\mathcal{N} = 1$ multiplets
	bosonic fields	fermionic fields	
1	metric $g_{\mu\nu}$	gravitino ψ_μ, ψ_μ^*	gravity multiplet
$i = 1, \dots, b_3(Y)$	scalars (S^i, P^i)	spinors χ^i, χ^{*i}	chiral multiplets Φ^i
$I = 1, \dots, b_2(Y)$	vectors A_μ^I	gauginos λ_α^I	vector multiplets V^I

Table 2.1: This table summarizes the massless four-dimensional low-energy effective $\mathcal{N} = 1$ supergravity spectrum that is obtained from the dimensional reduction of M-theory — or rather of eleven-dimensional supergravity — on a smooth G_2 -manifold Y .

on Y are in one-to-one correspondence with non-trivial cohomology elements of both $H^3(Y)$ and $H^2(Y)$,¹¹ and we arrive at the expansion of the four-dimensional chiral fermions in the zero modes

$$\chi(x)\zeta^{(1)}(y) = \sum_{i=1}^{b_3(Y)} \chi^i(x) \rho_{i,(nm)}^{\text{sym}} dy^n \otimes \gamma^m \eta + \sum_{I=1}^{b_2(Y)} \lambda^I \omega_{I[nm]}^{(2)} dy^n \otimes \gamma^m \eta, \quad (2.21)$$

in terms of the bases of zero modes ρ_i^{sym} of the Lichnerowicz Laplacian and of the harmonic two-forms $\omega_I^{(2)}$.

Now, we can spell out the massless four-dimensional spectrum in terms of $\mathcal{N} = 1$ supergravity multiplets as obtained from the dimensional reduction of M-theory upon the G_2 -manifolds Y . It consists of the four-dimensional supergravity multiplet, $b_3(Y)$ (neutral) chiral multiplets Φ^i , and $b_2(Y)$ (Abelian) vector multiplets V^I , as summarized in detail in Table 2.1.

To specify the four-dimensional low-energy effective $\mathcal{N} = 1$ supergravity action for the determined spectrum of the massless fields, we insert the mode expansions for the metric (2.7), the anti-symmetric three-form tensor (2.15), and the gravitino (2.16) into the eleven-dimensional supergravity action [14], which in terms of the eleven-

¹¹A priori, the constructed zero modes furnish elements of $H_1^3(Y)$, $H_{27}^3(Y)$ and $H_{14}^2(Y)$ that transform in the specified representations of the G_2 -structure group. However, on G_2 -manifolds all non-trivial three- and two-form cohomology elements can respectively be represented in the representations **1**, **27**, and **14**, which justifies the identification of zero modes with $H^3(Y)$ and $H^2(Y)$ [7], c.f. also Appendix A.

dimensional Hodge star $*_{11}$ and the eleven-dimensional gamma matrices $\hat{\Gamma}^M$ reads

$$\begin{aligned}
S_{11d} = & \frac{1}{2\kappa_{11}^2} \int \left(*_{11} \hat{R}_S - \frac{1}{2} d\hat{C} \wedge *_{11} d\hat{C} - *_{11} i \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNP} \hat{D}_N \hat{\Psi}_P \right) \\
& - \frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS} \hat{\Psi}_N (d\hat{C})_{[PQRS]} - \frac{1}{2\kappa_{11}^2} \int d\hat{C} \wedge *_{11} \hat{F} \\
& - \frac{1}{12\kappa_{11}^2} \int d\hat{C} \wedge d\hat{C} \wedge C + \dots, \quad (2.22)
\end{aligned}$$

where we denoted $\hat{F}_{[MNPQ]} = 3\bar{\hat{\Psi}}_{[M} \hat{\Gamma}_{NP} \hat{\Psi}_{Q]}$. The first line contains the kinetic terms of the eleven-dimensional supergravity multiplet, i.e., the Einstein–Hilbert term in terms of the Ricci scalar \hat{R}_S , the kinetic term for the anti-symmetric three-form tensor \hat{C} , and the Rarita–Schwinger kinetic term for the gravitino $\hat{\Psi}$. The second line comprises the interaction terms and the third line is the Chern–Simons term of the eleven-dimensional supergravity action. There are additional four-fermion interactions denoted by ‘...’ [14]. The coupling constant κ_{11} relates to the eleven-dimensional Newton constant \hat{G}_N , the eleven-dimensional Planck length $\hat{\ell}_P$ and Planck mass \hat{M}_P according to

$$\kappa_{11}^2 = 8\pi\hat{G}_N = \frac{(2\pi)^8 \hat{\ell}_P^9}{2} = \frac{(2\pi)^8}{2\hat{M}_P^9}. \quad (2.23)$$

To perform the Kaluza–Klein reduction let us introduce the moduli-dependent volume $V_Y(S^i)$ of the G_2 -manifold Y given by

$$V_Y(S^i) = \int_Y d^7y \sqrt{\det g(S^i)_{mn}}. \quad (2.24)$$

Furthermore, we introduce a reference G_2 -manifold Y_0 with respect to some background expectation values $S_0^i = \langle S^i \rangle$, upon which we carry out the dimensional reduction. This allows us to introduce the dimensionless (but yet moduli-dependent) volume factor

$$\lambda_0(S^i) = \frac{V_Y(S^i)}{V_{Y_0}} = \frac{1}{7} \int_Y \varphi \wedge *_{g_\varphi} \varphi, \quad (2.25)$$

in terms of the reference volume $V_{Y_0} = V_Y(S_0^i)$. Here the choice of Y_0 fixes via the resulting volume factor V_{Y_0} the normalization of the three-form φ .

Then — using eqs. (2.7) and (2.15) — the dimensional reduction of the Einstein–Hilbert term and the three-form tensor \hat{C} yields the four-dimensional action [17]

$$\begin{aligned}
S_{4d}^{\text{bos}} = & \frac{1}{2\kappa_4^2} \int \left[*_4 R_S + \frac{\kappa_{IJk}}{2V_{Y_0}} (S^k F^I \wedge *_4 F^J - P^k F^I \wedge F^J) \right. \\
& \left. - \frac{7}{2V_{Y_0}} \int_Y \rho_i^{(3)} \wedge *_{g_\varphi} \rho_j^{(3)} (dP^i \wedge *_4 dP^j - dS^i \wedge *_4 dS^j) \right]. \quad (2.26)
\end{aligned}$$

in terms of the four-dimensional Hodge star $*_4$, the Ricci scalar R_S with respect to the metric $g_{\mu\nu}$, the reference volume V_{Y_0} , and the seven-dimensional Hodge star $*_7$. Here we have performed the Weyl rescaling of the four-dimensional metric according to

$$g_{\mu\nu} \rightarrow \frac{g_{\mu\nu}}{\lambda_0(S^i)} , \quad (2.27)$$

such that the four-dimensional coupling constant κ_4 — relating to the four-dimensional Newton constant G_N , the four-dimensional Planck length ℓ_P and the Planck mass M_P — becomes

$$\kappa_4^2 = \frac{\kappa_{11}^2}{V_{Y_0}} , \quad \kappa_4^2 = 8\pi G_N = 8\pi\ell_P^2 = \frac{8\pi}{M_P^2} . \quad (2.28)$$

Furthermore, the couplings κ_{IJk} arise from the topological intersection numbers

$$\kappa_{IJk} = \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} , \quad (2.29)$$

and we have also used the following identification

$$\partial_i \partial_j \log \lambda_0 + \partial_i \log \lambda_0 \partial_j \log \lambda_0 = \frac{2}{7\lambda_0} \int_Y \rho_i^{(3)} \wedge *_{{g_\varphi}} \rho_j^{(3)} . \quad (2.30)$$

We can now bring the (bosonic) action (2.26) into the conventional form of four-dimensional $\mathcal{N} = 1$ supergravity [51].¹² To identify the chiral multiplets — that is to say to identify the complex structure of the Kähler target space — we observe that — at least to the leading order — the action of the membrane instantons generating non-perturbative superpotential interactions is given by [22]

$$\phi^i = P^i + iS^i . \quad (2.31)$$

Hence, due to holomorphy of the $\mathcal{N} = 1$ superpotential, the complex fields ϕ^i furnish complex coordinates of the Kähler target space and thus represent the complex scalar fields in the $\mathcal{N} = 1$ chiral multiplets Φ^i in Table 2.1. This allows us to readily read off from the action (2.26) the Kähler potential and the gauge kinetic coupling matrix [17, 18]

$$K(\phi, \bar{\phi}) = -3 \log \left(\frac{1}{7} \int_Y \varphi \wedge *_{{g_\varphi}} \varphi \right) , \quad (2.32)$$

$$f_{IJ}(\phi) = 2V_{Y_0} \sum_k \phi^k \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} = 2V_{Y_0} \sum_k \kappa_{IJk} \phi^k . \quad (2.33)$$

¹²The structure of the four-dimensional effective $\mathcal{N} = 1$ action from type II and M-theory dimensional reduction including Kaluza–Klein modes — which we do not consider here — has recently been discussed in refs. [52, 53].

Note that the holomorphy of the gauge kinetic coupling matrix is in accordance with the complex chiral coordinates (2.31). The moduli space metric is then given by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \frac{1}{2V_{Y_0} \lambda_0^2} \int_Y \rho_i^{(3)} \wedge *_{g_\varphi} \rho_{\bar{j}}^{(3)} . \quad (2.34)$$

Thus we see that in the physical theory the real scalar fields S^i and P^i combine to the complex chiral scalars ϕ^i according to eq. (2.31). These complex scalar fields parametrize locally the (semi-classical) M-theory moduli space $\mathcal{M}_{\mathbb{C}}$ of the G_2 compactification on Y of complex dimension $b_3(Y)$, where the real subspace $\text{Re}(\phi^i) = 0$ of real dimension $b_3(Y)$ is the geometric moduli space \mathcal{M} of G_2 -metrics on Y .¹³ Note, however, that the derived moduli space $\mathcal{M}_{\mathbb{C}}$ merely arises from the semi-classical dimensional reduction of eleven-dimensional supergravity on the G_2 -manifold Y . For the resulting four-dimensional $\mathcal{N} = 1$ supersymmetric theory, one expects on general grounds that the flat directions of $\mathcal{M}_{\mathbb{C}}$ are lifted at the quantum level due to non-perturbative effects in M-theory [22] — even in the absence of background fluxes.

Finally, let us remark that the presence of non-trivial four-form background fluxes G of anti-symmetric three-form tensor fields \hat{C} supported on the G_2 -manifold Y generates a flux-induced superpotential [16, 17, 20]. While the superpotential enters quadratically in the bosonic action, it appears linearly in the fermionic action generating a gravitino mass term M_ψ [51]

$$\mathcal{L}_{4d}^{M_\psi} = \frac{1}{2\kappa_4^2} e^{K/2} (\bar{W}(\bar{\phi}) \psi_\mu^T \gamma^{\mu\nu} \psi_\nu + W(\phi) \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu^*) . \quad (2.35)$$

This linear dependence on W allows us to directly derive the superpotential from the dimensional reduction of the gravitino terms, as carried in detail in Appendix A.4, where we determine the holomorphic superpotential to be¹⁴

$$W(\phi^i) = \frac{1}{4V_{Y_0}} \int_Y G \wedge (C + i\varphi) . \quad (2.36)$$

Our result is in agreement with the derivation of the flux-induced superpotential in refs. [16, 17, 20, 54]. Note that — both in the presence and in the absence of background fluxes G — we expect generically additional non-perturbative superpotential contributions arising from membrane instanton effects [4, 22].

¹³For corrections to the semi-classical moduli space $\mathcal{M}_{\mathbb{C}}$ see ref. [54].

¹⁴Non-vanishing four-form fluxes induce a gravitational back-reaction to the eleven-dimensional metric. This requires a warped metric ansatz [16] that breaks supersymmetry [2, 16]. As the presented derivation neglects such back-reactions, the resulting effective action becomes more accurate the smaller the effect of warping. Similarly as argued in ref. [20] this is the case for a small number of four-form flux quanta.

3 Kovalev's construction of G_2 -manifolds

In this section we focus on the construction of G_2 -manifold as put forward by Kovalev [23] and further developed by Corti et. al. [27, 24]. These G_2 -manifolds are obtained from a certain twisted connected sum of two asymptotically cylindrical Calabi–Yau threefolds times an additional circle S^1 . In the Kovalev limit the Ricci-flat metric of the obtained G_2 -manifold can be approximated by the metrics of the two Calabi–Yau summands. In order to set the stage for the derivation of the low-energy effective action for M-theory compactified on such G_2 -manifolds, we identify the Kovalev limit in the moduli space of the constructed G_2 -manifolds.

3.1 Asymptotically cylindrical Calabi–Yau threefold

A (complex) three-dimensional Calabi–Yau cylinder X^∞ is the product of a compact Calabi–Yau twofold — which we take to be a compact K3 surface S — with an open cylinder Δ^{cyl} , here given as the complement of the unit disk in the complex plane \mathbb{C} , i.e., $\Delta^{\text{cyl}} = \{z \in \mathbb{C} \mid |z| > 1\}$. The Kähler form ω^∞ and the holomorphic three-form Ω^∞ of X^∞ read

$$\begin{aligned}\omega^\infty &= \gamma^{*2} \frac{iz \wedge d\bar{z}}{2z\bar{z}} + \omega_S = \gamma^{*2} dt \wedge d\theta^* + \omega_S , \\ \Omega^\infty &= -\gamma^* \frac{iz}{z} \wedge \Omega_S = \gamma^* (d\theta^* - idt) \wedge \Omega_S ,\end{aligned}\tag{3.1}$$

in terms of the Kähler form ω_S and the holomorphic two-form Ω_S of the K3 surface S , the complex coordinate $z = e^{t+i\theta^*}$ and the length scale γ^* of the cylinder Δ^{cyl} .¹⁵ Then — with the metric g_S of the K3 surface — the product metric g_{X^∞} of the Calabi–Yau cylinder X^∞ becomes

$$g_{X^\infty} = \gamma^{*2} (dt^2 + d\theta^{*2}) + g_S .\tag{3.2}$$

The length scale γ^* furnishes the radius of the cylindrical metric on Δ^{cyl} , whereas the map $\xi : X^\infty \rightarrow \mathbb{R}^+$ with $\xi = \log |z|$ projects on the longitudinal direction of the cylinder, such that $\xi^{-1}(\mathbb{R}^+) = X^\infty$.

As defined in refs. [23, 27, 30] an asymptotically cylindrical Calabi–Yau threefold X is a non-compact Calabi–Yau threefold with $SU(3)$ holonomy and a complete Calabi–Yau metric g_X with the following properties. There exists a compact subspace $K \subset X$ such that the complement $X \setminus K$ is diffeomorphic to a three-dimensional Calabi–Yau cylinder X^∞ and such that the Kähler and holomorphic three-form of X approach fast enough ω^∞ and Ω^∞ of the cylindrical Calabi–Yau threefold X^∞ , as given in eqs. (3.1).

¹⁵With the conventional mutual normalization $(-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega} = \frac{\omega^n}{n!}$ between the Kähler form ω and the holomorphic n -form Ω of Calabi–Yau n -folds, we note that — assuming this normalization for ω_S and Ω_S of the K3 surface S — the Kähler form ω^∞ and the holomorphic three-form Ω^∞ of X^∞ are conventionally normalized.

More precisely, given the diffeomorphism $\eta : X^\infty \rightarrow X \setminus K$ we require that in the limit $\xi \rightarrow +\infty$ and for any positive integer k [23, 27, 30]

$$\begin{aligned} \eta^* \omega - \omega^\infty &= d\mu \quad \text{with} \quad |\nabla^k \mu| = O(e^{-\lambda \gamma^* \xi}) , \\ \eta^* \Omega - \Omega^\infty &= d\nu \quad \text{with} \quad |\nabla^k \nu| = O(e^{-\lambda \gamma^* \xi}) , \end{aligned} \quad (3.3)$$

for certain choices of μ and ν with the norm $|\cdot|$ and Levi-Civita connection ∇ of the metric g_{X^∞} . The scale λ has inverse length dimension and is determined by the (inverse) length scale of the asymptotic region X^∞ . To be precise [23]

$$\lambda = \min \left\{ \frac{1}{\gamma^*}, \lambda_S \right\} , \quad (3.4)$$

where λ_S is the square root of the smallest positive eigenvalue of the Laplacian of the K3 surface S in the asymptotic Calabi-Yau cylinder X^∞ .

3.2 Kovalev's twisted connected sum

From any Calabi-Yau threefold X with $SU(3)$ holonomy we can always construct the seven-manifold $X \times S^1$ with the torsion-free G_2 -structure

$$\varphi_0 = \gamma d\theta \wedge \omega + \text{Re}(\Omega) , \quad * \varphi_0 = \frac{1}{2} \omega^2 - \gamma d\theta \wedge \text{Im}(\Omega) , \quad (3.5)$$

in terms of the Kähler form ω , holomorphic three-form Ω of X and the coordinate θ of S^1 with radius γ . However, the resulting seven-dimensional manifold with φ_0 still has $SU(3)$ holonomy, which is a subgroup of G_2 .

For Kovalev's construction of the G_2 -manifolds Y [23], the essential idea is to first construct two suitable asymptotically cylindrical Calabi-Yau threefolds $X_{L/R}$ (referred to as left and right). Then for each of them one takes a direct product with $S^1_{L/R}$ in order to obtain two seven-manifolds $Y_{L/R}$ with torsion-free G_2 -structures $\varphi_{0L/R}$ and $SU(3)$ holonomy as above. To obtain a genuine compact G_2 -manifold Y , the asymptotic regions of type $Y^\infty_{L/R} = X^\infty_{L/R} \times S^1_{L/R}$ are glued together in such a way that the obtained manifold Y admits a torsion-free G_2 -structure resulting in G_2 holonomy and not a subgroup thereof.

To obtain the full G_2 holonomy, it is necessary to reduce the infinite fundamental groups $\pi_1(Y_{L/R})$ to a finite fundamental group $\pi_1(Y)$ by gluing their asymptotic regions appropriately. For suitable choices of $Y_{L/R}$ this can be achieved by Kovalev's twisted connected sum construction [23]. Recall that the asymptotic regions of $Y_{L/R}$ are given by

$$Y^\infty_{L/R} = X^\infty_{L/R} \times S^1_{L/R} = S_{L/R} \times \Delta^{\text{cyl}}_{L/R} \times S^1_{L/R} , \quad (3.6)$$

with the K3 surfaces $S_{L/R}$ and the cylinders $\Delta^{\text{cyl}}_{L/R}$ (c.f., Section 3.1). Let us denote by $\omega^I_{L/R}, \omega^J_{L/R}$ and $\omega^K_{L/R}$ the triplets of mutually orthogonal hyper Kähler two-forms,

satisfying the relations $(\omega_{L/R}^I)^2 = (\omega_{L/R}^J)^2 = (\omega_{L/R}^K)^2$. Then the $SU(2)$ structures of the asymptotic polarized K3 surfaces $S_{L/R}$ determine their Kähler two-forms $\omega_{S_{L/R}}^\infty$ and their holomorphic two-forms $\Omega_{S_{L/R}}^\infty$ according to

$$\omega_{S_{L/R}}^\infty = \omega_{L/R}^I, \quad \Omega_{S_{L/R}}^\infty = \omega_{L/R}^J + i\omega_{L/R}^K. \quad (3.7)$$

With eqs. (3.1) and (3.5) this explicitly specifies the asymptotic torsion-free G_2 -structures

$$\begin{aligned} \varphi_{0L/R}^\infty = & \gamma_{L/R} d\theta_{L/R} \wedge \left(\gamma_{L/R}^{*2} dt_{L/R} \wedge d\theta_{L/R}^* + \omega_{S_{L/R}}^\infty \right) \\ & + \gamma_{L/R}^* d\theta_{L/R}^* \wedge \text{Re}(\Omega_{S_{L/R}}^\infty) + \gamma_{L/R}^* dt \wedge \text{Im}(\Omega_{S_{L/R}}^\infty). \end{aligned} \quad (3.8)$$

Following Kovalev [23], let us now assume that the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ are chosen such that the resulting asymptotic polarized K3 surfaces $S_{L/R}$ are mutually isometric with respect to a hyper Kähler rotation $r : S_L \rightarrow S_R$ obeying¹⁶

$$r^* \omega_R^I = \omega_L^J, \quad r^* \omega_R^J = \omega_L^I, \quad r^* \omega_R^K = -\omega_L^K. \quad (3.9)$$

Then there is a family of diffeomorphisms $F_\Lambda : Y_L^\infty \rightarrow Y_R^\infty$ with constant $\Lambda \in \mathbb{R}$ given by

$$F_\Lambda : (\theta_L^*, t_L, u_L^\alpha, \theta_L) \mapsto (\theta_R^*, t_R, u_R^\alpha, \theta_R) = (\theta_L, \Lambda - t_R, r(u_L^\alpha), \theta_L^*), \quad (3.10)$$

in terms of the local coordinates $(\theta_{L/R}^*, t_{L/R})$ of $\Delta_{L/R}^{\text{cyl}}$, $u_{L/R}^\alpha$ of $S_{L/R}$, and $\theta_{L/R}$ of $S_{L/R}^1$. Now it is straightforward to check that if and only if the radii are equal

$$\gamma := \gamma_L = \gamma_R = \gamma_L^* = \gamma_R^*, \quad (3.11)$$

this asymptotic diffeomorphism is also an asymptotic isometry because it leaves the asymptotic G_2 -structures $\varphi_{0L/R}^\infty$ — and hence the asymptotic metric — invariant, i.e.,

$$F_\Lambda^* \varphi_{0R} = \varphi_{0L}. \quad (3.12)$$

As schematically depicted in Figure 3.1, the G_2 -manifold Y is now obtained by gluing the asymptotic ends of $Y_{L/R}$ with the help of the diffeomorphism F_Λ . Let $X_{L/R}(T)$ and $Y_{L/R}(T)$ be the truncated asymptotically cylindrical Calabi–Yau threefolds and the truncated seven manifolds given by cutting off their asymptotic regions at $t_{L/R} = T + 1$ for some (large) T , i.e.,

$$X_{L/R}(T) = K_{L/R} \cup \eta_{L/R}(\mathbb{R}_{<T+1}), \quad Y_{L/R}(T) = X_{L/R}(T) \times S_{L/R}^1. \quad (3.13)$$

¹⁶These conditions impose rather non-trivial constraints on the pair of asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$, which — at least for certain classes of pairs — have been studied systematically in ref. [24]. In this section we assume that these conditions on $X_{L/R}$ are met, and we come back to this issue in Section 5.3, where we explicitly construct asymptotically cylindrical Calabi–Yau threefold $X_{L/R}$ fulfilling these constraints.

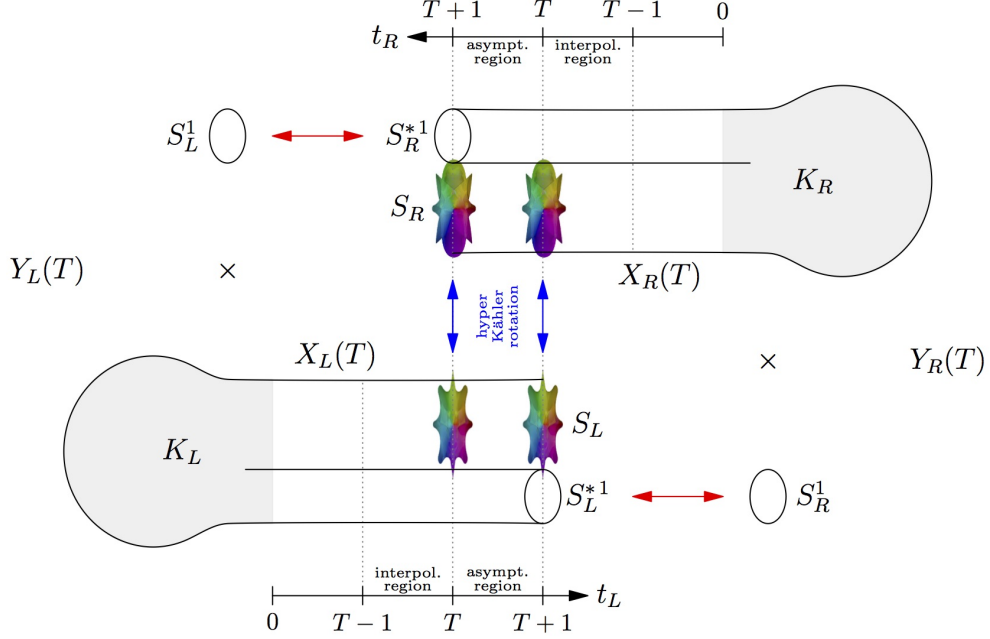


Figure 3.1: This picture illustrates Kovalev’s twisted connected sum. $X_{L/R}(T)$ are the truncated asymptotically cylindrical Calabi–Yau threefolds together with their compact subspaces $K_{L/R}$. Their Cartesian products with the circles $S_{L/R}^1$ yield the two seven-dimensional components $Y_{L/R}(T)$ combined to form the G_2 -manifold Y . There are two essential aspects in the gluing procedure. Firstly — as indicated by the red horizontal arrows — the circles $S_{L/R}^1$ are identified with the asymptotic circles of $X_{L/R}(T)$ here denoted by $S_{R/L}^{*1}$. Secondly — as depicted by the blue vertical arrows — the asymptotic polarized K3 surfaces $S_{L/R}$ must be matched with a certain hyper Kähler rotation. Finally, the diagram highlights the interpolating regions, $t_{L/R} \in (T-1, T]$, and the asymptotic gluing regions, $t_{L/R} \in (T, T+1)$, important for the construction of the G_2 -structure $\varphi(\gamma, T)$ of Y .

Here the diffeomorphisms $\eta_{L/R}$ and the compact subspaces $K_{L/R}$ are defined in Section 3.1. Then, using the (restricted) diffeomorphism F_{2T+1} — which maps the coordinate $t_L \in (T, T+1)$ to $t_R = -t_L + 2T + 1 \in (T, T+1)$ — we glue the two seven manifolds $Y_{L/R}(T)$ at the overlap $t_{L/R} \in (T, T+1)$ to arrive at the compact seven-manifold

$$Y = Y_L(T) \cup_{F_{2T+1}} Y_R(T) . \quad (3.14)$$

Finally, to construct a G_2 -structure φ on Y , we first introduce interpolating G_2 -structures on the two pieces $Y_{L/R}(T)$. Let $\alpha : \mathbb{R} \rightarrow [0, 1]$ be a smooth function interpolating between 0 and 1 within the interval $(-1, 0)$, namely $\alpha(s) = 0$ for $s \leq -1$ and $\alpha(s) = 1$ for $s \geq 0$. Then we endow the truncated asymptotically cylindrical

Calabi–Yau threefolds $X_{L/R}$ with the forms [23, 27]

$$\begin{aligned}\tilde{\omega}_{L/R}^T &= \omega_{L/R} - d(\alpha(t - T)\mu_{L/R}) , \\ \tilde{\Omega}_{L/R}^T &= \Omega_{L/R} - d(\alpha(t - T)\nu_{L/R}) ,\end{aligned}\tag{3.15}$$

in terms of the forms $\mu_{L/R}$ and $\nu_{L/R}$ of eqs. (3.3). By construction the forms $\tilde{\omega}_{L/R}^T$ and $\tilde{\Omega}_{L/R}^T$ smoothly interpolate between the corresponding Calabi–Yau cylinder forms (3.1) and the asymptotically cylindrical Calabi–Yau forms (3.3). At the interpolating regions $t_{L/R} \in (T - 1, T)$ the symplectic forms $\omega_{L/R}^T$ fail to induce a Ricci-flat metric and the three-forms $\Omega_{L/R}^T$ cease to be holomorphic. Analogously to eq. (3.5), the interpolating G_2 -structures $\tilde{\varphi}_{L/R}(\gamma, T)$ on $Y_{L/R}$ read

$$\tilde{\varphi}_{L/R}(\gamma, T) = \gamma d\theta \wedge \tilde{\omega}_{L/R}^T + \text{Re}(\tilde{\Omega}_{L/R}^T) ,\tag{3.16}$$

which according to eq. (3.12) glue together to a well-defined G_2 -structure $\tilde{\varphi}(\gamma, T)$ on the seven-manifold Y .

Note that the constructed G_2 -structure $\tilde{\varphi}(\gamma, T)$ is closed but not torsion-free. The torsion of $\varphi(\gamma, T)$ is measured by $d*\tilde{\varphi}(\gamma, T)$. It is only non-vanishing at the interpolating regions $t_{L/R} \in (T - 1, T)$, where it is of order $O(e^{-\gamma\lambda T})$ due to eq. (3.3) [23].¹⁷ Hence, it is plausible that we can view $\tilde{\varphi}(\gamma, T)$ as an order $O(e^{-\gamma\lambda T})$ approximation to a torsion-free G_2 -structure $\varphi(\gamma, T)$, which equips the seven-manifold Y with a Ricci-flat metric. Indeed, Kovalev shows that for sufficiently large T there exists in the same three-form cohomology class of $\tilde{\varphi}(\gamma, T)$ a torsion-free G_2 -structure $\varphi(\gamma, T)$, such that for any positive integer k [23]

$$\varphi(\gamma, T) = \tilde{\varphi}(\gamma, T) + d\rho(\gamma, T) \quad \text{with} \quad |\nabla^k \rho(\gamma, T)| = O(e^{-\gamma\lambda T}) ,\tag{3.17}$$

in terms of the norm $|\cdot|$ and the Levi–Civita connection ∇ of the metric induced from the asymptotic G_2 -structure (3.12).

Finally, the relationship $\pi_1(Y) = \pi_1(X_L) \times \pi_1(X_R)$ among the fundamental groups in the twisted connected sum implies that the torsion-free G_2 -structure $\varphi(\gamma, T)$ indeed gives rise to a genuine G_2 -manifold Y with G_2 holonomy [23].

This summarizes the main result of ref. [23] — clarified and further developed in refs. [24, 30, 27] — namely the analysis proof that the G_2 -structure $\tilde{\varphi}(\gamma, T)$ in Kovalev’s twisted connected sum construction furnishes an approximation to the torsion-free G_2 -structure $\varphi(\gamma, T)$ in the same three-form cohomology class, which gives rise to a compact seven-dimensional Ricci-flat Riemannian manifold Y with G_2 holonomy.

¹⁷Note that — due to relation (3.11) and the hyper Kähler isometry (3.12) — we have the identifications $\lambda = \lambda_L = \lambda_R$ among inverse length scales.

3.3 The Kovalev limit of G_2 -manifolds

As discussed in the last section the torsion-free G_2 -structure $\varphi(\gamma, T)$ in Kovalev's twisted connected sum is approximated via eq. (3.5) in terms of the holomorphic forms $\Omega_{L/R}$ and the Kähler forms $\omega_{L/R}$ of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$. According to eqs. (3.3) and (3.17) this approximation is of order $O(e^{-\gamma\lambda T})$.

We now want to discuss the torsion-free G_2 -structure $\varphi(\gamma, T)$ as a function of the parameters γ and T . Except for the overall volume modulus, we keep all other moduli of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ fixed. That is to say, we consider the moduli dependence of the two metrics $g_{L/R}$ of $X_{L/R}$ as

$$g_{L/R}(z_{L/R}, t_{L/R}) = \gamma_0^2 R^2 \tilde{g}(z_{L/R}, \tilde{t}_{L/R}) , \quad (3.18)$$

where $z_{L/R}$ and $t_{L/R}$ are the (dimensionless) complex structure moduli and Kähler moduli of $X_{L/R}$, respectively. The constant γ_0 has dimension of length, such that the metrics $\tilde{g}_{L/R}$ becomes dimensionless. We split the Kähler moduli $t_{L/R}$ further into the overall volume modulus R and the remaining Kähler moduli $\tilde{t}_{L/R}$ — measuring ratios of volumes of subvarieties in $X_{L/R}$ — such that¹⁸

$$t_{L/R}^a = \begin{cases} R^2 & a = 1 \\ R^2 \tilde{t}_{L/R}^a & a \neq 1 \end{cases} . \quad (3.19)$$

In order to obtain the G_2 -manifold Y from the seven-dimensional building blocks $Y_{L/R}$, the hyper Kähler compatibility condition (3.9) constrains the explicit values of the moduli $z_{L/R}$ and $\tilde{t}_{L/R}$. Furthermore, the required identification (3.11) of the radii of all circles in the asymptotic region of $Y_{L/R}$ determines the volume modulus R as the dimensionless ratio

$$R = \frac{\gamma}{\gamma_0} , \quad (3.20)$$

and it justifies to introduce a mutual volume modulus R for both threefolds $X_{L/R}$.

In the Kovalev limit of large RT the volume V_Y of the constructed G_2 -manifold Y becomes

$$V_Y(\tilde{S}, R, T) = V_{Y_L(T)}(z_L, t_L) + V_{Y_R(T)}(z_R, t_R) - (2\pi)^2 \gamma_0^3 R^3 V_S(\tilde{\rho}_S, R) + O(e^{-\tilde{\lambda}RT}) . \quad (3.21)$$

The resulting volume depends on the moduli R and T and the remaining moduli of the G_2 -manifold Y — collective denoted by \tilde{S} singling out those moduli fields $\tilde{\rho}_S$ deforming the K3 surface S . The volumes $V_{Y_{L/R}(T)}$ of the truncated building blocks $Y_{L/R}(T)$

¹⁸For an asymptotically cylindrical Calabi–Yau threefold with a single Kähler modulus t the volume modulus R relates to this Kähler modulus as $R^2 = t$ without the presence of any further moduli \hat{t} .

are calculated with the metrics of the (truncated) asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}(T)$ using the expressions (3.1) and the volume formula (2.25) (with the dimensionful reference volume $V_{Y_0} = \gamma_0^7$). As the sum of the first two terms counts the volume of the overlapping region twice, we need to subtract this contribution again once. It is given by the product of the volumes of the overlapping interval, the asymptotic two-torus $S_L^1 \times S_L^{*1} \equiv S_R^1 \times S_R^{*1}$, and the asymptotic K3-surface $S_L \equiv S_R \equiv S$ according to

$$V_{Y_L(T) \cap Y_R(T)} = (2\pi)^2 \gamma_0^3 R^3 V_S(\tilde{\rho}_S, R) = (2\pi)^2 \gamma_0^7 R^7 V_S^{\tilde{g}}(\tilde{\rho}_S) . \quad (3.22)$$

In the last equality the volume of the K3-surface S is expressed in terms of the dimensionless (asymptotic) metric \tilde{g} . Due to the approximation (3.3) of the metrics of the building blocks $Y_{L/R}(T)$ by the limiting metrics of $Y_{L/R}^\infty(T)$ in the interpolation region $t_{L/R} \in (T-1, T)$ and due to the overall correction (3.17) to the torsion free G_2 -structure $\varphi(\gamma, T)$, for large RT the computed volume of the G_2 -manifold Y receives exponentially suppressed corrections in $\tilde{\lambda}RT$, where — because of eq. (3.4) — the dimensionless constant $\tilde{\lambda}$ reads $\tilde{\lambda} = \lambda\gamma_0$.

Note that — up to the exponentially correction terms suppressed for large RT — the volume (3.22) is entirely determined by the (relative) periods and the Kähler forms of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$. However, due to the non-compactness of $X_{L/R}$ the relative periods and the Kähler volumes of non-compact cycles diverge linearly in T . Therefore, in order to extract the required geometric data from the diverging periods and infinite volumes a suitable regularization scheme must be employed in order to extract the finite periods and finite volumes of the truncated asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}(T)$. This analysis is beyond the scope of this work, but we plan to get back to this issue elsewhere.

Thus, instead of deriving the entire moduli dependence of the volume of the G_2 -manifold Y , we focus on the moduli dependence of the two fields R and T — viewing the remaining moduli fields \tilde{S} as parameters. First, we compute the volumes $V_{Y_{L/R}(T)}$ in eq. (3.22) as

$$\begin{aligned} V_{Y_{L/R}(T)} &= \frac{\gamma_0^7}{7} \int_{Y_{L/R}(T)} \varphi_{L/R} \wedge * \varphi_{L/R} \\ &= \frac{\gamma_0^7}{7} \int_{Y_{L/R}(T) \setminus K_{L/R}} \varphi_{L/R}^\infty \wedge * \varphi_{L/R}^\infty + V_{K_{L/R}} + (2\pi)^2 \gamma_0^7 R^7 \Delta_{L/R}(\tilde{S}, T) \\ &= (2\pi)^2 \gamma_0^7 R^7 \left[V_S^{\tilde{g}}(\tilde{\rho}_S) \left((T+1) + D_{L/R}(\tilde{S}) \right) + \Delta_{L/R}(\tilde{S}, T) \right] . \end{aligned} \quad (3.23)$$

Here we split the integration by performing the integral over the compact parts $K_{L/R}$ and the asymptotic regions $Y_{L/R}(T) \setminus K_{L/R}$. The former part factors into the volume of the K3 surface S and a contribution $D_{L/R}(\tilde{S})$ to be discussed in greater detail momentarily. The latter part is evaluated with respect to the asymptotic G_2 -structure

$\varphi_{L/R}^\infty$, which introduces the correction term $\Delta_{L/R}(\tilde{S}, T)$ such that

$$\Delta_{L/R}(\tilde{S}, T) = R \int_0^{T+1} dt f_{L/R}(\tilde{S}, Rt) e^{-\tilde{\lambda} Rt} = C_{L/R}(\tilde{S}) + O(e^{-\tilde{\lambda} RT}) , \quad (3.24)$$

in terms of the function $f_{L/R}(\tilde{S}, Rt)$ determined by eq. (3.3). Thus, taking the correction terms $\Delta_{L/R}$ into account we arrive at

$$V_Y(\tilde{S}, R, T) = (2\pi)^2 \gamma_0^7 R^7 V_S^{\tilde{g}}(\tilde{\rho}_S) \left(2T + \alpha(\tilde{S}) \right) + O(e^{-\tilde{\lambda} RT}) , \quad (3.25)$$

with

$$\alpha(\tilde{S}) = \left(D_L(\tilde{S}) + C_L(\tilde{S}) \right) + \left(D_R(\tilde{S}) + C_R(\tilde{S}) \right) + 1 . \quad (3.26)$$

As discussed the moduli-dependent contributions $D_{L/R}(\tilde{S}) + C_{L/R}(\tilde{S})$ — and hence the moduli dependent function $\alpha(\tilde{S})$ — are in principle computable from the (relative) periods and the Kähler forms of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$.

In the context of M-theory compactification on the G_2 -manifold Y the volume V_Y determines the four-dimensional Planck constant κ_4 according to eq. (2.28). We refer to the Kovalev limit as the approximation in which the four-dimensional Planck constant κ_4 — and hence the volume V_Y — remains constant, while the exponential correction terms become sufficiently small. Namely, for fixed moduli \tilde{S} we require that the dimensionless quantity χ , that is given by

$$\chi^7 = R^7(2T + \alpha) , \quad (3.27)$$

remains constant. Requiring a constant four-dimensional Planck constant yields for the modulus R the functional dependence

$$R(T) = \frac{\chi}{\sqrt[7]{2T + \alpha}} , \quad (3.28)$$

such that corrections terms scale as

$$O(e^{-\tilde{\lambda} RT}) = O(e^{-\frac{\tilde{\lambda} \chi T}{\sqrt[7]{2T + \alpha}}}) . \quad (3.29)$$

Thus, for large T with $R = R(T)$ — as in eq. (3.28) — the corrections in eq. (3.25) for the volume V_Y are exponentially suppressed. Thus, for $T \gg \tilde{\lambda} \chi$ — loosely referred to as the Kovalev limit — the torsion-free G_2 -structure of Y is well approximated in terms of the geometric data of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ for a given four-dimensional Planck constant κ_4 . However, taking literally the limit $T \rightarrow \infty$ with $R = R(T)$ does not yields a limiting Riemannian manifold but instead yields only a Hausdorff limit in the sense of Gromov–Hausdorff convergence of compact metric spaces. In the context of M-theory the Kovalev limit implies that

— while for large T with $R = R(T)$ the Ricci-flat G_2 -metric gets more and more accurately approximated in terms of the Ricci-flat Calabi–Yau metrics of $X_{L/R}$ — the discussed semi-classical dimensional reduction on the G_2 -manifold Y in terms of the Kaluza–Klein zero modes becomes less accurate due to the emergence of both light Kaluza–Klein modes and substantial non-perturbative membrane and M5-brane instanton corrections.

4 M-theory on twisted connected sums

In this section we analyze the four-dimensional low-energy effective action of M-theory compactifications on G_2 -manifolds that are of the twisted connected sum type. We focus on the Kovalev limit, in which the asymptotically cylindrical Calabi–Yau metrics of the summands furnish a good approximation to the G_2 -metric. Our first task is to analyze the four-dimensional $\mathcal{N} = 1$ spectrum of such compactifications, which — according to Table 2.1 — amounts to expressing the de Rham cohomology groups of the resulting G_2 -manifolds in terms of the cohomology groups of the Calabi–Yau summands. In particular, we explicitly identify the chiral modulus governing the Kovalev limit, referred to as the Kovalevton \varkappa , and we discuss the effective action and its physical properties in this limit.

4.1 Spectrum and cohomology

To deduce the four-dimensional $\mathcal{N} = 1$ supersymmetric spectrum of M-theory compactified on a G_2 -manifold of the twisted connected sum type, we want to analyze the de Rham cohomology of Y as arising from the cohomology of the asymptotically cylindrical Calabi–Yau summands. A partial answer to this question has already been given in Kovalev’s paper [23]. In ref. [24] Corti et. al. have presented a systematic analysis of the cohomology of Y , which we summarize and use here.

Following refs. [27, 24], let us first introduce the notion of a building block (Z, S) , which allows us to construct an asymptotically cylindrical Calabi–Yau threefold X . In this work a building block (Z, S) is a pair consisting of a smooth K3 fibration $\pi : Z \rightarrow \mathbb{P}^1$ together with a smooth K3 fiber $S = \pi^{-1}(p)$ for some $p \in \mathbb{P}^1$. We require that the anti-canonical class $-K_Z$ is primitive and that S is linearly equivalent to the anti-canonical class, i.e., $S \sim -K_Z$. Due to the fibrational structure, the self-intersection of S is trivial, which implies that the manifold $X = Z \setminus S$ has the topology of an asymptotically cylindrical Calabi–Yau threefold that admits a Ricci-flat Kähler metric [23, 27, 24].¹⁹ As in refs. [27, 24], we further impose two technical assumptions on the

¹⁹This can readily be seen as follows: The trivial normal bundle of S in X defines a tubular neighborhood $T_\epsilon(S) \subset X$. By construction $T_\epsilon^*(S) := T_\epsilon(S) \setminus S$ is homeomorphic to $\Delta^{\text{cyl}} \times S$, which has the topology of a three-dimensional Calabi–Yau cylinder X^∞ to be viewed as the asymptotic region of the asymptotically cylindrical Calabi–Yau threefold X , as discussed in Section 3.1.

building block (Z, S) . Namely, we demand that $H^3(Z, \mathbb{Z})$ is torsion-free and that the integral two-form cohomology $H^2(X, \mathbb{Z})$ embeds primitively into the K3 lattice $L = H^2(S, \mathbb{Z})$ via the pullback map of the inclusion $\rho : S \hookrightarrow X$ (which is well-defined up to homotopy).

To construct a G_2 -manifold Y we now consider a pair of building blocks $(Z_{L/R}, S_{L/R})$ such that the polarized K3 surfaces $S_{L/R}$ are isometric and fulfill the hyper Kähler matching condition $r : S_L \rightarrow S_R$. From the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R} = Z_{L/R} \setminus S_{L/R}$ we can then construct the G_2 -manifold Y with Kovalev’s twisted connected sum construction as detailed in Section 3.2. Under the assumptions in the definition of the building blocks $(Z_{L/R}, S_{L/R})$, Corti et. al. derive the cohomology of the G_2 -manifold Y from the building blocks [24]

$$\begin{aligned} \pi_1(Y) &= H^1(Y, \mathbb{Z}) = 0 , \\ H^2(Y, \mathbb{Z}) &\simeq (k_L \oplus k_R) \oplus (N_L \cap N_R) , \\ H^3(Y, \mathbb{Z}) &\simeq H^3(Z_L, \mathbb{Z}) \oplus H^3(Z_R, \mathbb{Z}) \oplus k_L \oplus k_R \oplus N_L \cap T_R \oplus N_R \cap T_L \\ &\quad \oplus \mathbb{Z}[S] \oplus L / (N_L + N_R) . \end{aligned} \tag{4.1}$$

Here $[S]$ is the Poincaré dual three-form of a K3 fiber S in the building blocks $(Z_{L/R}, S_{L/R})$, and L denotes the K3 lattice $L \simeq H^2(S_L, \mathbb{Z}) \simeq H^2(S_R, \mathbb{Z})$. Furthermore, the inclusion maps $\rho_{L/R} : S_{L/R} \hookrightarrow X_{L/R}$ induce the maps $\rho_{L/R}^* : H^2(X_{L/R}, \mathbb{Z}) \rightarrow L$, which define the kernels $k_{L/R} := \ker \rho_{L/R}^*$, the images $N_{L/R} := \text{Im } \rho_{L/R}^*$, and the transcendental lattices

$$T_{L/R} = N_{L/R}^\perp = \{l \in L \mid \langle l, N_{L/R} \rangle = 0\} . \tag{4.2}$$

Note that — by the assumptions imposed on the cohomological properties of the buildings blocks — the images $N_{L/R}$ (and hence the sum $N_L + N_R$) are primitive sublattices of the K3 lattice L . As a consequence the quotient $L/(N_L + N_R)$ is torsion-free, and — due to the assumed torsion-freeness of $H^3(Z_{L/R}, \mathbb{Z})$ — all the integral cohomology groups (4.1) are torsion-free as well.

Thus, the properties of the building blocks determine the cohomology (4.1) and therefore the effective four-dimensional $\mathcal{N} = 1$ supersymmetric spectrum of the M-theory compactification according to Table 2.1. Let us now in particular focus on the neutral chiral moduli multiplets Φ^i associated to the three-form cohomology $H^3(Y, \mathbb{Z})$. The derivation of the cohomology in ref. [24] is essentially based upon the Mayer–Vietoris sequence applied to the union (3.14) of the overlapping non-compact seven manifolds $Y_{L/R}$ defined in eq. (3.13) in terms of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$, such that

$$\begin{aligned} H^3(Y, \mathbb{Z}) &= \ker \left(H^3(Y_L, \mathbb{Z}) \oplus H^3(Y_R, \mathbb{Z}) \xrightarrow{(\iota_L^*, -\iota_R^*)} H^3(T^2 \times S, \mathbb{Z}) \right) \\ &\quad \oplus \text{coker} \left(H^2(Y_L, \mathbb{Z}) \oplus H^2(Y_R, \mathbb{Z}) \xrightarrow{(\iota_L^*, -\iota_R^*)} H^2(T^2 \times S, \mathbb{Z}) \right) , \end{aligned} \tag{4.3}$$

where $Y_R \cap Y_L$ deformation retracts to $T^2 \times S$ and the maps are induced from the standard inclusion maps $\iota_{L/R} : T^2 \times S \hookrightarrow Y_{L/R}$. The three-form cohomology (4.1) is distributed among these two summands in the following way [24]

$$\begin{aligned} \ker (H^3(Y_L, \mathbb{Z}) \oplus H^3(Y_R, \mathbb{Z}) \rightarrow H^3(T^2 \times S, \mathbb{Z})) \\ = H^3(Z_L, \mathbb{Z}) \oplus H^3(Z_R, \mathbb{Z}) \oplus k_L \oplus k_R \oplus N_L \cap T_R \oplus N_R \cap T_L , \quad (4.4) \\ \text{coker} (H^2(Y_L, \mathbb{Z}) \oplus H^2(Y_R, \mathbb{Z}) \rightarrow H^2(T^2 \times S, \mathbb{Z})) = \mathbb{Z}[S] \oplus L / (N_L + N_R) . \end{aligned}$$

Expanding the torsion-free G_2 -structure φ of Y in terms of the above assembled three-form cohomology basis as in eq. (2.14), the coefficients of the individual cohomology elements capture (in the Kovalev limit) particular geometric moduli of the twisted connected sum and their summands.

First of all, the kernel contributions (4.4) describe the moduli of the asymptotically cylindrical Calabi–Yau manifolds $X_{L/R}$. In particular, the coefficients of $H^3(Z_{L/R})$ and $k_{L/R}$ respectively realize complex structure moduli and Kähler moduli of the asymptotically cylindrical Calabi–Yau manifolds $X_{L/R}$. Furthermore, $N_L \cap T_R$ captures mutual Kähler moduli of X_L and complex structure moduli of X_R , which are inter-linked in this way due to the non-trivial gluing with the hyper Kähler rotation (3.9), and — exchanging X_L and X_R — $N_R \cap T_L$ enjoys the analog interpretation.

Second of all, we analyze the cokernel contribution in eq. (4.4). To get a better geometric picture for these moduli, we first observe that — due to the K3 fibrations $Z_{L/R} \rightarrow \mathbb{P}^1$ — the Calabi–Yau threefolds $X_{L/R}$ are K3 fibrations over a disk $D_{L/R}$. As a result $Y_{L/R}$ become K3 fibrations over solid tori $T_{L/R} \equiv S^1_{L/R} \times D_{L/R}$, namely

$$\begin{array}{ccc} S_{L/R} & \longrightarrow & Y_{L/R} \\ & & \downarrow \pi \\ & & T_{L/R} . \end{array} \quad (4.5)$$

The gluing diffeomorphism (3.10) in the twisted connected sum identifies the boundary of the disk D_L with the circle S^1_R and the circle S^1_L with the boundary of the disk D_R such that the two solid tori $T_{L/R}$ are glued together to a three-sphere S^3 . Thus, the resulting G_2 -manifold Y is a topological K3 fibration²⁰

$$\begin{array}{ccc} S & \longrightarrow & Y \\ & & \downarrow \pi \\ & & S^3 . \end{array} \quad (4.6)$$

The cohomology three-forms of the cokernel (4.4) describe moduli of the asymptotic boundary of $\partial Y_L \simeq \partial Y_R \simeq T^2 \times S$. Their dual homology three-cycles restrict to

²⁰The topological K3 fibration in the context of twisted connected sums has also been discussed in ref. [26].

relative three-cycles in the summands Y_L and Y_R , and hence the associated moduli are sensitive to the overlapping gluing regions $Y_{L/R}(T) \setminus K_{L/R}$, c.f., Figure 3.1. In particular, the three-form generator $[S]$ is Poincaré dual to a K3 fiber S , and hence its dual homology three-cycles is the S^3 base of the fibration (4.6). As a consequence the modulus associated to $[S]$ measures the volume of the base S^3 . Similarly, the remaining cokernel moduli measure volumes of three-cycles that project under the map $\pi : Y \rightarrow S^3$ to paths in the S^3 base connecting the disjoint compact subsets $\pi(K_L)$ and $\pi(K_R)$ of S^3 . Note that $2T + 1$ is the distance between these two compact subsets in terms of the parameter T introduced in Section 3.3. Therefore, it now follows that — in the Kovalev limit — all cokernel moduli depend linearly on the parameter T , and geometrically T enjoys the interpretation of a squashing parameter for the S^3 base of the K3 fibration (4.6).

The split into two types of moduli fields in eq. (4.4) motivates us to introduce two universal geometric moduli v and b . For any G_2 -manifold there is the universal volume modulus v that is associated to the singlet $H_1^3(Y, \mathbb{Z})$ of the three-form cohomology. It simply rescales the torsion-free G_2 -structure φ . In the twisted connected sum we additionally identify the squashing modulus b of the S^3 base in the fibration (4.6). Note that $b \rightarrow +\infty$ describes the Kovalev limit discussed in Section 3.3. The torsion-free G_2 -structure φ depends according to eq. (2.14) on these two moduli as

$$\varphi(v, b, \tilde{S}) = v \left[\left(\rho_0^{\text{ker}} + \sum_i \tilde{S}^i \rho_i^{\text{ker}} \right) + b \left([S] + \sum_i \tilde{S}^i \rho_i^{\text{coker}} \right) \right]. \quad (4.7)$$

Here $[S]$ is the harmonic three-form that is Poincaré dual to the K3 fiber S . Furthermore, $(\rho_0^{\text{ker}}, \rho_i^{\text{ker}})$ and ρ_i^{coker} form a basis of harmonic three-forms arising from the kernel contributions (4.4) and the cokernel part $L/(N_L + N_R)$, respectively. \tilde{S}^i and \tilde{S}^i are the respective associated geometric real moduli fields.²¹

Thus the description of the torsion free G_2 -structure $\varphi(v, b, \tilde{S})$ gives rise to two universal $\mathcal{N} = 1$ neutral chiral moduli multiplets ν and \varkappa given by

$$\text{Re}(\nu) = v, \quad \text{Re}(\varkappa) = vb. \quad (4.8)$$

In particular, we refer to the chiral multiplet \varkappa as the Kovalevton, as it describes in the limit $\text{Re}(\varkappa) \rightarrow +\infty$ — while keeping $\text{Re}(\nu)$ constant — the Kovalev limit discussed in Section 3.3.

The remaining real moduli fields are not universal and relate to the non-universal neutral chiral multiplets as

$$\text{Re}(\phi^i) = v\tilde{S}^i, \quad \text{Re}(\phi^{\tilde{i}}) = vb\tilde{S}^{\tilde{i}}. \quad (4.9)$$

²¹Note that the kernel contribution (4.4) is at least one-dimensional, such that we can always choose a basis element ρ_0^{ker} [24].

They depend on the topological details of the building blocks $(Z_{L/R}, S_{L/R})$ and the choice of gluing diffeomorphism (3.10).

Finally, the two-form cohomology $H^2(Y, \mathbb{Z})$ for (smooth) G_2 -manifolds yields four-dimensional massless abelian $\mathcal{N} = 1$ vector multiplets, c.f., Table 2.1. In Kovalev's twisted connected sum we get two types of $\mathcal{N} = 1$ vector multiplets according to eq. (4.1), which we discuss in the sequel.

Firstly, the kernel contributions k_L and k_R associate to zero modes of the two summands Y_L and Y_R . Thus, we can view $Y_{L/R}$ as the local geometries governing these gauge theory degrees of freedom. As the individual summands $Y_{L/R} = S^1_{L/R} \times X_{L/R}$ have $SU(3)$ holonomy in the Kovalev limit, we expect that the two gauge theory sectors of the kernels k_L and k_R exhibit $\mathcal{N} = 2$ supersymmetry. Indeed — in addition to the abelian $\mathcal{N} = 1$ vector multiplets — the kernels $k_{L/R}$ of the local geometries $Y_{L/R}$ also contribute to the three-form cohomology $H^3(Y, \mathbb{Z})$ resulting in $\mathcal{N} = 1$ neutral chiral multiplets. Thus, the abelian $\mathcal{N} = 1$ vector and the neutral $\mathcal{N} = 1$ chiral multiplets associated to the $k_{L/R}$ readily combine into four-dimensional $\mathcal{N} = 2$ vector multiplets.

Secondly, in the Kovalev limit the abelian vector multiplets obtained from the intersection $N_L \cap N_R$ can be attributed to the local geometry of the asymptotic regions $Y_L(T) \cap Y_R(T) \simeq T^2 \times S \times (0, 1)$, which has $SU(2)$ holonomy. Thus, we expect that these vector multiplets give rise to a four-dimensional abelian $\mathcal{N} = 4$ gauge theory sector, which can be seen as follows. To any two-form $\omega^{(2)}$ in $N_L \cap N_R$ of the K3 surface S , we attribute the three-forms

$$\omega^{(2)} \wedge h(t) d\theta_L, \quad \omega^{(2)} \wedge h(t) d\theta_R, \quad \omega^{(2)} \wedge h(t) dt, \quad (4.10)$$

in terms of the coordinates $\theta_{L/R}$ of $S^1_L \times S^1_R \simeq T^2$ and the smooth bump function $h(t)$ in the coordinate t of the interval $(0, 1)$.²² These three-forms yield geometrically non-trivial cohomology elements of compact support in $H^3_c(T^2 \times S \times (0, 1), \mathbb{Z})$, which give rise to normalizable scalar fields. They combine with three scalar deformations of the hyper Kähler metric of the K3 surface S to three complex scalar moduli fields, which furnish three neutral four-dimensional $\mathcal{N} = 1$ chiral multiplets. It is these three $\mathcal{N} = 1$ chiral multiplets that combine with the $\mathcal{N} = 1$ vector multiplet of $\omega^{(2)}$ to one $\mathcal{N} = 4$ vector multiplet.²³ Note that the three-forms (4.10) canonically extend to Kovalev's G_2 -manifold Y . However, they become trivial in cohomology because $N_L \cap N_R$ is not

²²The bump function $h(t)$ is given by a smooth non-negative function $h(t) : (0, 1) \rightarrow \mathbb{R}$ with compact support, which is normalized such that $\int_0^1 h(t) dt = 1$.

²³Alternatively, we can consider the five-dimensional theory obtained from M-theory on $T^2 \times S$ with $SU(2)$ holonomy. Then the two-form cohomology element $\omega^{(2)}$ is accompanied by the two three-form cohomology elements $\omega^{(2)} \wedge d\theta_{L/R}$. Combined with the mentioned three hyper Kähler metric deformations these cohomology elements provide for the zero modes of five scalar fields, which — together with the vector field and the superpartners — assemble into a five-dimensional $\mathcal{N} = 2$ vector multiplet for each harmonic two-form $\omega^{(2)}$. Upon dimensional reduction to four dimensions, we arrive at four-dimensional $\mathcal{N} = 4$ vector multiplets.

local geometry (Kovalev limit)	multiplicity of $\mathcal{N} = 1$ multiplets		$U(1)$ vector multiplets	
	$U(1)$ vectors	chirals	multiplicity	supersym.
$Y_L = S_L^1 \times X_L$ $SU(3)$ holonomy	$\dim k_L$	$\dim k_L$	$\dim k_L$	$\mathcal{N} = 2$
$Y_R = S_R^1 \times X_R$ $SU(3)$ holonomy	$\dim k_R$	$\dim k_R$	$\dim k_R$	$\mathcal{N} = 2$
$T^2 \times S \times (0, 1)$ $SU(2)$ holonomy	$\dim N_L \cap N_R$	$3 \cdot \dim N_L \cap N_R$	$\dim N_L \cap N_R$	$\mathcal{N} = 4$

Table 4.1: Shown are the abelian gauge theory sectors of the local geometries appearing in twisted connected sum G_2 -manifolds in the Kovalev limit $T \rightarrow +\infty$. The left column specifies the Ricci-flat local geometries with their holonomies. The middle column lists the $\mathcal{N} = 1$ chiral multiplets that assemble in the right column to vector multiplets of extended supersymmetry.

an element of $H^3(Y, \mathbb{Z})$ according to eq. (4.1). Nevertheless, we can Fourier expand any of these three-forms into eigenforms with respect to the three-form Laplacian Δ of the G_2 -manifold Y . By a simple scaling argument we find that the eigenvalues of these three-form Fourier modes scale with T^{-1} , i.e., they are inversely proportional to the parameter T realizing the Kovalev limit. Therefore, we argue that the normalizable zero modes associated to the three-forms (4.10) acquire a mass term $m^2 \simeq O(T^{-1})$, which vanishes in the Kovalev limit. Furthermore, we expect that the scalar fields associated to the hyper Kähler metric deformations are generically obstructed at first order by a mass term that also vanishes in the Kovalev limit. As a consequence, we deduce that the massless four-dimensional abelian $\mathcal{N} = 4$ vector multiplets of the asymptotic region decomposes into a massless four-dimensional abelian $\mathcal{N} = 1$ vector multiplet and three massive four-dimensional $\mathcal{N} = 1$ chiral multiplets with masses of order $O(T^{-1/2})$. Thus, we expect that the four-dimensional $\mathcal{N} = 4$ gauge theory sector is only realized in the strict Kovalev limit $T \rightarrow +\infty$.

The discussed local abelian gauge theory sectors are summarized in Table 4.1. In particular, we find that in the Kovalev limit — at least in the absence of background four-form fluxes and for smooth G_2 -manifolds Y — the spectrum of all abelian gauge theory sectors exhibit extended supersymmetry. The observed extended supersymmetries of the local geometries appearing in Kovalev’s twisted connected sum become relevant in the Section 5 and Section 6, because they impose strong constraints on the non-Abelian gauge theory sectors with charged matter fields.

4.2 The Kähler potential

The aim of this subsection is to describe the universal properties of the four-dimensional low energy effective action in terms of the universal chiral multiplets ν and \varkappa . We first establish that — while keeping the ratio $\text{Re}(\nu)/\text{Re}(\varkappa)$ constant — the chiral multiplet ν directly relates to the (dimensionless) volume modulus R of Section 3.3 as

$$\text{Re}(\nu) = R^3 . \quad (4.11)$$

This relation comes about because the $\text{Re}(\nu)$ measures (dimensionless) volumes of three-cycles while R measures (dimensionless) length scales in the G_2 -manifold Y . Apart from the overall volume dependence, the Kovalevton \varkappa measures the squashed volume of the S^3 base. Therefore, from the expression (3.25) of the volume $V_Y(\tilde{S}, R, T)$ we arrive at the relation

$$\text{Re}(\varkappa) = (2\pi)^2 R^3 (2T + \alpha(\tilde{S})) , \quad (4.12)$$

where \tilde{S} denotes collectively the remaining geometric moduli fields $\tilde{S}^{\tilde{i}}$ and $\tilde{S}^{\tilde{i}}$.

Thus — using eqs. (2.32) and (3.25) — we find that the universal structure of the four-dimensional low energy effective $\mathcal{N} = 1$ supergravity action is governed by the Kähler potential

$$K(\nu, \bar{\nu}, \varkappa, \bar{\varkappa}) = -4 \log(\nu + \bar{\nu}) - 3 \log(\varkappa + \bar{\varkappa}) - 3 \log \left(V_S^{\tilde{g}}(\tilde{S}) \right) . \quad (4.13)$$

Note that this Kähler potential is only a valid approximation both in the large volume regime and in the Kovalev regime, where quantum corrections and metric corrections of the asymptotically cylindrical Calab–Yau threefolds are suppressed. The semi-classical large volume limit arises for both $\text{Re}(\nu)$ and $\text{Re}(\varkappa)$ to be taken sufficiently large, while the corrections to the G_2 -metric in the twisted connected sum are suppressed if in addition $\text{Re}(\varkappa)$ is even (parametrically) larger than $\text{Re}(\nu)$, c.f., the discussion at the end of Section 3.3.

Let us discuss some basic properties of the derived Kähler potential. First of all, the structure of the Kähler potential is reminiscent of the Kovalev limit, in which the volume of the G_2 -manifold is dominated by the cylindrical region $S \times T^2 \times I$ in terms of the interval I of size $2T + 1$. That is to say in this limit the individual summands in eq. (4.13) reflect the volume of the K3 surface S , the squashed volume of the S^3 base dominated by $T^2 \times I$, and the moduli dependence of the K3 fiber S on the non-universal moduli \tilde{S} . As long as we treat the non-universal moduli fields \tilde{S} as constants, the Kähler geometry for the universal Kähler moduli ν and \varkappa factorizes into two (complex) one-dimensional parts with a block diagonal Kähler metric. However, this block structure in the Kähler metric vanishes as soon as we treat the non-universal moduli fields \tilde{S} dynamically, because relation (4.9) implies that the real geometric moduli \tilde{S} also depend non-trivially on the chiral fields ν and

\varkappa as

$$\tilde{S}^i = \frac{\phi^i + \bar{\phi}^i}{\nu + \bar{\nu}} , \quad \tilde{S}^{\bar{i}} = \frac{\phi^{\bar{i}} + \bar{\phi}^{\bar{i}}}{\varkappa + \bar{\varkappa}} . \quad (4.14)$$

We now briefly discuss the Kähler potential (4.13) with the non-universal moduli fields \tilde{S} treated as constants. We first observe that

$$g^{i\bar{j}} \partial_i K \partial_{\bar{j}} K - 3 = 4 \geq 0 , \quad i \in \{\nu, \varkappa\}, \bar{j} \in \{\bar{\nu}, \bar{\varkappa}\} , \quad (4.15)$$

in terms of the inverse Kähler metric $g^{i\bar{j}}$. This implies that the no-scale inequality $g^{i\bar{j}} \partial_i K \partial_{\bar{j}} K - 3 \geq 0$ is fulfilled (but not saturated). The no-scale inequality is a property of the Kähler potential only and it guarantees that the scalar potential of the described four-dimensional $\mathcal{N} = 1$ supergravity theory is positive semi-definite for any non-vanishing superpotential [55]. As a result the analyzed Kähler potential (of the two chiral fields ν and \varkappa only) does not admit a negative cosmological constant and hence no (supersymmetric) anti-de-Sitter vacua.

Finally, we record the Kähler potential including the leading order correction to the Kovalev limit, which according to eq. (3.25) takes the form

$$K = -\log \left[\left(V_{\tilde{S}}^{\tilde{g}}(\tilde{S}) \right)^3 (\nu + \bar{\nu})^4 (\varkappa + \bar{\varkappa})^3 + A(\tilde{S}, \nu + \bar{\nu}, \varkappa + \bar{\varkappa}) e^{-\lambda \frac{\varkappa + \bar{\varkappa}}{(\nu + \bar{\nu})^{1/3}}} \right] , \quad (4.16)$$

where the coefficient of the exponentially suppressed correction is expected to generically depends on both universal and non-universal geometric moduli fields. A detailed analysis of this class of Kähler potential may exhibit interesting phenomenological properties, which, is, however, beyond the scope of this work.

5 Twisted connected sums from orthogonal gluing

In this section we discuss the explicit construction of twisted connected sum G_2 -manifolds by the method of orthogonal gluing [24]. This construction offers a systematic way to fulfill the matching condition (3.9) for a pair of asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ that are obtained from building blocks $(Z_{L/R}, S_{L/R})$ associated to semi-Fano threefolds $P_{L/R}$. We focus on building blocks $(Z_{L/R}, S_{L/R})$ of polarized K3 surfaces $S_{L/R}$ with Picard lattices of low rank and generate a list of new examples in order to get an impression of the multitude of possibilities to realize twisted connected sum G_2 -manifolds in terms of orthogonal gluing.

Our motivation for studying the method of orthogonal gluing is also to pan out the possibilities to obtain gauge theory sectors in twisted connected sum G_2 -manifolds. In Section 4 we have established that in the Kovalev limit this spectrum of vector multiplets assembles itself into $\mathcal{N} = 2$ and $\mathcal{N} = 4$ sectors as summarized in Table 4.1. While we postpone the analysis of the $\mathcal{N} = 2$ sectors to the next section, the focus in this section is on the $\mathcal{N} = 4$ gauge theory sectors.

In the context of the orthogonal gluing construction a certain intersection lattice R determines the $\mathcal{N} = 4$ gauge theory sector. In particular, the rank of this intersection lattice becomes the rank of the gauge group. We further argue that in the $\mathcal{N} = 4$ gauge theory sector the method of orthogonal gluing does not admit enhancements to non-Abelian gauge groups. As a result, we therefore arrive in the Kovalev limit at four-dimensional Abelian $\mathcal{N} = 4$ gauge theory sectors with gauge group $U(1)^{\text{rk } R}$.

5.1 Asymptotically cylindrical Calabi–Yau examples

Following ref. [27], we obtain from toric weak Fano threefolds P a rich class of building blocks (Z, S) , which in turn give rise to asymptotically cylindrical Calabi–Yau threefolds as discussed in Section 4.1. A projective smooth threefold P is weak Fano if its anti-canonical divisor $-K_P$ is nef and big, which means that the intersections obey $-K_P C \geq 0$ for any algebraic curve C in P and $(-K_P)^3 > 0$.²⁴ Assuming further that two global sections s_0 and s_1 of the anti-canonical divisor $-K_P$ intersect transversely in a smooth reduced curve $\mathcal{C} = \{s_0 = 0\} \cap \{s_1 = 0\} \subset P$ and that $S = \{\alpha_0 s_0 + \alpha_1 s_1 = 0\} \subset P$ is a smooth K3 surface for some choice of $[\alpha_0 : \alpha_1] \in \mathbb{P}^1$, a building block (Z, S) is obtained from the blow up $\pi_{\mathcal{C}} : Z \rightarrow P$ along \mathcal{C} , i.e.,

$$Z = \text{Bl}_{\mathcal{C}} P = \{(x, z) \in P \times \mathbb{P}^1 \mid z_0 s_0 + z_1 s_1 = 0\} , \quad (5.1)$$

together with the proper transform S of the smooth anti-canonical divisor S on P [27]. Note that for ease of notation we use the same symbol S for both the K3 surface in the toric weak Fano threefold P and its proper transform in the blow-up Z . Then the K3 fibration $\pi : Z \rightarrow \mathbb{P}^1$ becomes, c.f., Section 4.1,

$$\pi : Z \rightarrow \mathbb{P}^1, (x, z) \mapsto z , \quad (5.2)$$

and $S = \pi^{-1}([\alpha_0, \alpha_1])$ is the K3 surface of the building block (Z, S) . Moreover, the three-form Betti number $b_3(Z)$ of the blown-up threefold Z becomes

$$b_3(Z) = b_3(P) + 2g(\mathcal{C}) = b_3(P) + (-K_P)^3 + 2 . \quad (5.3)$$

Here $g(\mathcal{C})$ denotes the genus of the curve \mathcal{C} and the last equality follows from the adjunction formula.

For twisted connected sum G_2 -manifolds, however, it is essential to find a pair of asymptotically cylindrical Calabi–Yau threefolds that fulfill the matching condition (3.9). A common strategy is to first focus only on the moduli spaces of the polarized K3 surfaces $S_{L/R}$, ignoring their origin from the building blocks $(Z_{L/R}, S_{L/R})$ [24, 27, 56]. Once a matching pair of polarized K3 surfaces $S_{L/R}$ is found, it is necessary to check if these particular K3 surfaces $S_{L/R}$ arise as zero sections of anti-canonical

²⁴A smooth projective variety is Fano if its anti-canonical divisor is ample, i.e., $-K_P C > 0$ for any algebraic curve C in P .

divisors in $Z_{L/R}$. Beauville's theorem guarantees that indeed any general $K3$ surface polarized by the anti-canonical divisor of a Fano threefold can be realized as the zero locus of a global anti-canonical section for a suitable choice of the Fano threefold in its moduli space [57]. Thus, for a pair of building blocks obtained from Fano threefolds it suffices to construct a matching pair of polarized $K3$ surfaces $S_{L/R}$ to ensure the existence of a pair of matching building blocks $(Z_{L/R}, S_{L/R})$ in the moduli spaces of these building blocks.

However, for building blocks $(Z_{L/R}, S_{L/R})$ obtained from weak Fano threefolds the procedure of simply matching their polarized $K3$ surfaces $S_{L/R}$ may not be sufficient. That is to say, in this more general setting the entire moduli space of the polarized $K3$ surfaces cannot necessarily be obtained from a global anti-canonical section within the associated family of weak Fano threefolds. Therefore, Corti et. al. introduce the notion of semi-Fano threefolds that furnish a subclass of weak Fano threefolds, for which Beauville's theorem is still applicable [24, 27]. For the rather technical definition of semi-Fano threefolds we refer to ref. [27], but instead we present a characterizing criterion for toric semi-Fano threefolds momentarily.

An important and large class of building blocks (Z, S) is constructed from toric weak Fano threefolds P_Σ described in terms of a three-dimensional toric fan Σ . We describe the toric fan Σ of a toric weak Fano threefold P_Σ in terms of a three-dimensional reflexive lattice polytope Δ spanned by the one-dimensional cones of Σ , together with a triangulation, which encodes the higher-dimensional cones of the fan Σ .²⁵ By the classification of Kreuzer and Skarke [60, 61] there are 4319 three-dimensional reflexive polytopes, which often admit several triangulations, i.e., typically of the order of ten to a few hundred triangulations. In this work we focus on the class of toric semi-Fano threefolds P_Σ in order to follow the outlined recipe to construct explicit matching pairs for twisted connected sum G_2 -manifolds. In the toric setting the semi-Fano threefolds P_Σ are characterized by those reflexive polytopes Δ that do not have any interior points inside co-dimension one faces [27]. Note that there are 899 three-dimensional reflexive polytopes of the semi-Fano type [27].

The toric approach to semi-Fano threefolds provides for a powerful combinatorial machinery to explicitly carry out computations. For instance, a general global section s_Δ of the anti-canonical line bundle $-K_{P_\Sigma}$ is readily described by

$$s_\Delta = \sum_{\nu_i \in \Delta} s_i \prod_{\nu_k^* \in \Delta^*} x_k^{\langle \nu_i, \nu_k^* \rangle + 1}, \quad (5.4)$$

with the points ν_i and ν_k^* of the lattice polytope Δ and its dual lattice polytope Δ^* , the duality lattice pairing $\langle \cdot, \cdot \rangle$, the toric homogenous coordinates x_k , and the coordinates s_i on the space of global anti-canonical sections. Furthermore, for the discussed three-dimensional semi-Fano lattice polytopes a choice of generic sections s_0 and s_1 yields a smooth reduced curve $\mathcal{C} = \{s_0 = 0\} \cap \{s_1 = 0\} \subset P_\Sigma$ with a smooth $K3$ surface

²⁵See, for instance, refs. [58, 59] for an introduction to toric geometry.

$S = \{\alpha_0 s_0 + \alpha_1 s_1 = 0\} \subset P_\Sigma$ such that the blow-up (5.1) together with the proper transform S yields indeed a well-defined building block (Z, S) , which in the following we refer to as the toric semi-Fano building block (Z, S) . Such building blocks exhibit the required technical properties that the anti-canonical class $-K_Z$ is primitive, that $H^3(Z, \mathbb{Z})$ is torsion-free (because $H^3(P_\Sigma, \mathbb{Z})$ is torsion-free), and that $H^2(S, \mathbb{Z})$ embeds primitively into $H^2(X, \mathbb{Z})$ with $X = Z \setminus S$, c.f., ref. [27].

5.2 Construction of G_2 -manifolds

To construct explicit examples of twisted connected sum G_2 -manifolds, Corti et. al. introduce in ref. [24] the method of orthogonal gluing, which is a particular recipe to fulfill the matching condition for suitable pairs of building blocks $(Z_{L/R}, S_{L/R})$.

In this work we apply the orthogonal gluing method mainly to Fano building blocks and toric semi-Fano building blocks to algorithmically find novel twisted connected sum G_2 -manifolds. In order to specify a particular semi-Fano threefold, we use in the following the Mori–Mukai classification for Fano threefold varieties [62] and the Kasprzyk classification for reflexive polytopes with terminal singularities for certain toric semi-Fano threefolds [63,64]. We label the corresponding semi-Fano threefolds by their respective reference numbers MM# $_\rho$ or/and K# in these classifications, where the subscript in the Mori–Mukai list denotes the rank ρ of the Picard lattice of the Fano threefold.

Applying the method of orthogonal gluing to a pair of building blocks $(Z_{L/R}, S_{L/R})$ of two semi-Fano threefolds $P_{L/R}$, we use the following three-step algorithm [24]:

- *Construction of the orthogonal pushout lattice W* : To achieve the matching condition (3.9) first for a pair of polarized K3 surfaces $S_{L/R}$ via orthogonal gluing, choose a negative definite lattice R embedded primitively into both Picard lattices $N_{L/R}$ of the polarized K3 surfaces $S_{L/R}$. Then the lattice W is constructed as

$$W = N_L + N_R, \quad R = N_L \cap N_R, \quad (5.5)$$

such that

$$N_L^\perp \subset N_R, \quad N_R^\perp \subset N_L, \quad (5.6)$$

with the orthogonal lattices $N_{L/R}^\perp$ defined in eq. (4.2). The lattice W is called the orthogonal pushout of $N_{L/R}$ with respect to R and is also denoted by [24]

$$W = N_L \perp_R N_R. \quad (5.7)$$

Note that the pushout lattice W is unique but in general need not exist because the non-degenerate lattice pairing $\langle \cdot, \cdot \rangle_W : W \times W \rightarrow \mathbb{Z}$ induced from the pairings $\langle \cdot, \cdot \rangle_{N_{L/R}} : N_{L/R} \times N_{L/R} \rightarrow \mathbb{Z}$ is not necessarily well-defined. That is to say the induced pairing

$$\langle e_L + e_R, f_L + f_R \rangle_W = \langle e_L, f_L \rangle_{N_L} + \langle e_R, f_R \rangle_{N_R}, \quad (5.8)$$

must be integral for any pair of lattice points $(e_L + e_R, f_L + f_R)$ in W , which can be presented — not necessarily uniquely — by $e_{L/R}, f_{L/R} \in N_{L/R}$. Furthermore, we require that the intersections $W_{L/R} = N_{L/R} \cap N_{R/L}^\perp$ with the (generic) Kähler cones $\mathcal{K}(P_{L/R})$ of (the deformation families of) $P_{L/R}$ are non-empty, i.e.,

$$\mathcal{K}(P_{L/R}) \cap W_{L/R} \neq \emptyset . \quad (5.9)$$

- *Primitive embedding of pushout lattice W into K3 lattice L :* The matching condition (3.9) for a suitable pair of polarized K3 surfaces $S_{L/R}$ in their moduli spaces is achieved if we embed primitively the pushout lattice W into the K3 lattice L . The existence of such an embedding can often be deduced from results by Nikulin [65]. In particular, such a primitive embedding is guaranteed to exist if the following rank condition is fulfilled [24]

$$\text{rk } N_L + \text{rk } N_R \leq 11 . \quad (5.10)$$

- *Lift matching condition of K3 surfaces $S_{L/R}$ to building blocks $(Z_{L/R}, S_{L/R})$:* Finally, we must ensure that the matching condition (3.9) for the polarized K3 surfaces $S_{L/R}$ can be achieved within the moduli space of the building blocks $(Z_{L/R}, S_{L/R})$. Corti et. al. show in Proposition 6.18 of ref. [24] that the imposed assumptions on the orthogonal pushout W — namely that $(Z_{L/R}, S_{L/R})$ are building blocks of semi-Fano threefolds, that the lattice R is negative definite, that the intersections (5.9) are non-empty, and that W embeds primitively into the K3 lattice L — are sufficient to ensure that the matching conditions of the polarized K3 surfaces $S_{L/R}$ can indeed be lifted to the moduli spaces of building blocks $(Z_{L/R}, S_{L/R})$.

Let us now determine the cohomology groups (4.1) of the G_2 -manifolds Y obtained from orthogonal gluing. First we observe that $\text{rk } N_L \cap T_R$ and $\text{rk } N_R \cap T_L$ equal $\text{rk } W_L$ and $\text{rk } W_R$, respectively, while $N_L + N_R$ becomes the orthogonal pushout W with $\text{rk } W = \text{rk } W_L + \text{rk } W_R + \text{rk } R$. Therefore, we readily deduce for the Betti numbers $b_2(Y) = \dim H^2(Y)$ and $b_3(Y) = \dim H^3(Y)$ for the G_2 -manifolds Y or the orthogonal gluing type

$$\begin{aligned} b_2(Y) &= \text{rk } R + \dim k_L + \dim k_R , \\ b_3(Y) &= b_3(Z_L) + b_3(Z_R) + \dim k_L + \dim k_R - \text{rk } R + 23 . \end{aligned} \quad (5.11)$$

Here $b_3(Z_{L/R})$ are the three-form Betti numbers of the threefolds $Z_{L/R}$ and $\dim k_{L/R}$ are the dimensions of the kernels $k_{L/R}$ defined below eq. (4.1).

Recall that in the Kovalev limit the kernels $k_{L/R}$ describe the $\mathcal{N} = 2$ gauge theory sectors, respectively, whereas the rank of the intersection lattice R in the orthogonal pushout W coincides with the rank of the gauge group of the $\mathcal{N} = 4$ gauge theory sector, c.f. Table 4.1. A particular simple choice of orthogonal gluing is achieved

if the intersection lattice R has rank zero, i.e., $N_L \cap N_R = \{0\}$. This special case of orthogonal gluing is referred to as perpendicular gluing with its trivial orthogonal pushout W denoted by [24]

$$W = N_L \perp N_R . \quad (5.12)$$

As a consequence, in the Kovalev limit the $\mathcal{N} = 4$ gauge theory sector is absent if the twisted connected G_2 -manifold is obtained via perpendicular gluing.

5.3 Examples of G_2 -manifolds from orthogonal gluing

In this section, we study concrete examples of twisted connected sum G_2 -manifolds obtained via orthogonal gluing. In particular, we focus on examples with non-trivial intersection lattices R in the orthogonal pushout W .

As explained in Section 4.1, in the Kovalev limit such examples yield $\mathcal{N} = 4$ gauge theory sectors with the Abelian gauge group $U(1)^{\text{rk } R}$. Enhancement to a non-Abelian gauge group G of rank r would occur if the intersection lattice $R = N_L \cap N_R$ had a sublattice $G(-1)$ or rank r with the pairing given by minus the Cartan matrix of the Lie algebra of G [66]. Then we could blow-down the mutual r rational curves of both polarized K3 surfaces $S_{L/R}$ because — by the definition of the intersection lattice $R = G(-1)$ resides in the intersection of both Picard lattices $N_{L/R}$. In this way we would arrive at a singular polarized K3 surfaces $S_{L/R}$ resulting in the enhanced $\mathcal{N} = 4$ gauge theory sector with non-Abelian gauge group $G \times U(1)^{\text{rk } R - r}$. However, using the method of orthogonal gluing such a gauge theory enhancement is not possible because the orthogonal complement to R in the polarized K3 surfaces $S_{L/R}$ is required to contain an ample class, which — due to the ampleness — would always have a non-zero intersection with any rational curve. It would nevertheless be interesting to see if non-Abelian gauge groups are possible in the $\mathcal{N} = 4$ gauge theory sector by generalizing the orthogonal gluing construction.

Orthogonal gluing of rank two semi-Fano threefolds: Let us consider orthogonal gluings among building blocks of semi-Fano threefolds with Picard number two. The only non-trivial orthogonal gluings among such building blocks — that is to say apart from perpendicular gluings — have an intersection lattice R of rank one. In ref. [56] Crowley and Nordström classify for rank two Fano threefold building blocks all possible non-trivial orthogonal gluings to twisted connected sum G_2 -manifolds — except for one missing pair.²⁶ As a warm-up we want to extend this classification by including all building blocks arising from toric semi-Fano threefolds with Picard number two.

²⁶In Table 2 and Table 4 of ref. [56] the authors list all rank two Fano building blocks and the resulting G_2 -manifolds, respectively. However, the classification of G_2 -manifolds in Table 4 misses the orthogonal gluing between the building blocks MM5₂ and MM25₂ of Table 2. Therefore, the Theorem 5.10 should enumerate nineteen instead of eighteen pairs of twisted connected sum G_2 -manifolds.

It turns out that there is actually a unique toric semi-Fano threefold P_Σ of Picard number two that is not Fano and is given by the projective bundle [27]

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)) \rightarrow \mathbb{P}^1 . \quad (5.13)$$

Its toric realization of P_Σ arises from the reflexive lattice polytope Δ and its dual reflexive polytope Δ^* spanned by the lattice points ν_1, \dots, ν_5 and the dual lattice points ν_1^*, \dots, ν_5^* given by:

$$\begin{array}{ll} \Delta : & \nu_1 = (-1, -1, 0) & \Delta^* : & \nu_1^* = (-1, -1, 1) \\ & \nu_2 = (1, 0, 0) & & \nu_2^* = (-1, 2, -2) \\ & \nu_3 = (0, 1, 0) & & \nu_3^* = (-1, 2, 1) \\ & \nu_4 = (1, 1, 1) & & \nu_4^* = (2, -1, -2) \\ & \nu_5 = (0, 0, -1) & & \nu_5^* = (2, -1, 1) \end{array} \quad (5.14)$$

The reflexive lattice polytope Δ appears as number K32 in the Kasprzyk classification [63, 64]. It admits two simplicial triangulations both realizing the projective bundle (5.13). The toric variety P_Σ associated to the fan Σ of one of these triangulations gives rise to the Mori cone spanned by the curves $C_B \simeq \mathbb{P}^1$ and the curve $C_F \simeq \mathbb{P}^1 \subset \mathbb{P}_F^2$ in a projective fiber \mathbb{P}_F^2 , such that these curves have the following intersection numbers with the toric divisors D_i associated to the vertices ν_i :²⁷

$$\begin{array}{rccccc} & D_1 & D_2 & D_3 & D_4 & D_5 \\ \hline C_F : & 1 & 1 & 1 & 0 & 0 \\ C_B : & 0 & -1 & -1 & 1 & 1 \end{array} \quad (5.15)$$

As a consequence, we find among the toric divisors the linear equivalences $D_2 \sim D_3$, $D_4 \sim D_5$, $D_2 \sim D_1 - D_4$ and the Kähler cone $\mathcal{K}(P_\Sigma)$ spanned by

$$\mathcal{K}(P_\Sigma) = \langle\langle D_1, D_4 \rangle\rangle . \quad (5.16)$$

Furthermore, note that $-K_{P_\Sigma} = \sum_i D_i \sim 3D_1$ and $-K_{P_\Sigma}^3 = 54$. The intersection matrix κ_{P_Σ} of the generators D_1 and D_4 with the anti-canonical class $-K_{P_\Sigma}$ reads

$$\kappa_{P_\Sigma} = \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix} , \quad (5.17)$$

and has discriminant $\Delta^\kappa = -9$. The intersection matrix κ_{P_Σ} furnishes the intersection pairing of the Picard lattice of the anti-canonical K3 surface in P_Σ .

²⁷For the other triangulation the Mori cone takes the form:

$$\begin{array}{rccccc} & D_1 & D_2 & D_3 & D_4 & D_5 \\ \hline C_F : & 1 & 0 & 0 & 1 & 1 \\ C_B : & 0 & 1 & 1 & -1 & -1 \end{array}$$

No.	$-K^3$	κ	Δ^κ	A	e	A^2	e^2	$b_3(Z)$
K32 (semi-Fano)	54	$\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$	-9	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 3 \end{pmatrix}$	12	-12	56
MM5 ₂ (Fano)	12	$\begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}$	-9	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 3 \\ -1 \end{pmatrix}$	12	-12	26
MM25 ₂ (Fano)	32	$\begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$	-16	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$	12	-12	34

Table 5.1: The rows of the table list the data of the unique rank two $\rho = 2$ toric semi-Fano threefolds together with all rank two Fano threefolds that admit an intersection lattice R generated by a vector of length square -12 . The columns show the reference numbers MM# _{ρ} or K# in the Mori–Mukai [62] or Kasprzyk [64] classification respectively, the triple intersection of the anti-canonical divisor $-K$, the intersection matrix κ of the Picard lattice of the anti-canonical K3 surface, the discriminant Δ^κ of the intersection matrix κ , the chosen ample class A in the basis of the intersection matrix κ , the orthogonal complement e to the ample class A , the length squares of the classes A and e , and the three-form Betti number $b_3(Z)$ of the associated building block Z .

According to the described algorithm in Section 5.2, for a pair of Picard lattices (N_L, N_R) of rank two to yield a non-trivial orthogonal pushout W , the rank one sublattices $W_{L/R}$ must be generated by ample classes in $\mathcal{K}(P_{L/R})$ with the intersection lattice R orthogonal to both $W_{L/R}$. Thus, we need to construct two ample classes $A_{L/R}$ together with orthogonal lattice vectors $e_{L/R}$ in $N_{L/R}$ with $e_L^2 = e_R^2$, which generate the rank one intersection lattice R . Then — as Crowley and Nordström show in ref. [56] — the induced lattice pairing $\langle \cdot, \cdot \rangle_W$ is a well-defined integral lattice pairing, if and only if

$$\frac{\Delta_L^\kappa \Delta_R^\kappa}{A_L^2 A_R^2} = k^2, \quad \text{for } k \in \mathbb{Z}, \quad (5.18)$$

in terms of the discriminants $\Delta_{L/R}^\kappa$ and the ample classes $A_{L/R}$. Moreover, in order to fulfill this matching condition with a rank two Fano threefold $P_{R/L}$, Crowley and Nordström deduce for the rank two semi-Fano threefold $P_{L/R}$ the upper bound [56]

$$\left| \frac{\Delta_{L/R}^\kappa}{A_{L/R}^2} \right| \leq \frac{8}{5}. \quad (5.19)$$

To find a matching rank two Fano building block for non-trivial orthogonal gluing with the unique rank two toric semi-Fano threefold (5.13), we first choose an ample class A , which according to eq. (5.16) is given by $A = nD_1 + mD_4$ with $A^2 = 6n(n+m)$. Thus, in order to conform with the inequality (5.19) for $\Delta^\kappa = -9$, the only possible ample class is $A = D_1 + D_4$ with $A^2 = 12$. For this class the orthogonal complement

R is generated by $e = -D_1 + 3D_4$ with $e^2 = -12$. Table 5.1 summarizes the data of this toric semi-Fano threefold together with the corresponding data for the building blocks of rank two Fano threefolds with compatible rank one intersection lattices generated by vectors of length square -12 . The latter entries are taken from the Crowley–Nordström classification [56].

For the entries in Table 5.1, condition (5.18) tells us the two possible gluings with rank one intersection lattices, namely

$$\begin{aligned} W_{\text{MM}25_2}^{\text{K}32} &= N_{\text{K}32} \perp_e N_{\text{MM}25_2} : & b_2(Y_{\text{MM}25_2}^{\text{K}32}) &= 1, & b_3(Y_{\text{MM}25_2}^{\text{K}32}) &= 112, \\ W_{\text{MM}25_2}^{\text{MM}5_2} &= N_{\text{MM}5_2} \perp_e N_{\text{MM}25_2} : & b_2(Y_{\text{MM}25_2}^{\text{MM}5_2}) &= 1, & b_3(Y_{\text{MM}25_2}^{\text{MM}5_2}) &= 82. \end{aligned} \quad (5.20)$$

Here $Y_{\text{MM}25_2}^{\text{K}32}$ denotes the twisted connected G_2 -manifolds obtained from the orthogonal pushout $W_{\text{MM}25_2}^{\text{K}32}$ with their Betti numbers computed by eq. (5.11). The first G_2 -manifold realizes the only possible combination with the rank two toric semi-Fano threefold K32, whereas the second G_2 -manifold realizes the non-trivial orthogonal gluing among those rank two Fano threefolds that has been overseen in ref. [56].

Orthogonal gluing of higher rank semi-Fano threefolds: As our next illustrating examples, we consider orthogonal gluings along a rank one intersection lattice with the rank three Fano threefold $P_L = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which has the reference numbers MM27₃ and K62 in the Mori–Mukai and Kasprzyk classifications [62–64]. Let H_i , $i = 1, 2, 3$, be the hyperplane classes of the respective \mathbb{P}^1 -factors of this Fano threefold, which generate the three-dimensional Kähler cone

$$\mathcal{K}(P_\Sigma) = \langle\langle H_1, H_2, H_3 \rangle\rangle. \quad (5.21)$$

The ample anti-cannonical class becomes $-K_{P_L} = 2H_1 + 2H_2 + 2H_3$, and the intersection matrix κ_{P_L} of the Kähler cone generators with the anti-canonical divisor $-K_{P_L}$ reads

$$\kappa_{P_L} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}. \quad (5.22)$$

Now, we focus on orthogonal gluing with the rank one intersection lattice generated by a vector e of length square $e^2 = -4$. Note that this length square realizes the maximal negative value, as the pairing κ_{P_L} is even and vectors with length square $e^2 = -2$ correspond to rational curves with a positive intersection number with any ample class A , which is in violation with the orthogonal gluing assumption (5.9).

The intersection pairing (5.22) corresponds to the ternary quadratic form $q(x, y, z) = 4(xy + yz + zx)$, which allows us to parametrize with the help of ref. [67] — up to trivial permutations of the Kähler cone generators H_1, H_2, H_3 — all vectors e with $e^2 = -4$ by

$$e = (d_1 - k)H_1 + (d_2 - k)H_2 + kH_3, \quad (5.23)$$

where the integers k, d_1, d_2 obey

$$k^2 - d_1 d_2 = 1, \quad 0 \leq d_1 < k \leq d_2. \quad (5.24)$$

In order to fulfill condition (5.9), we need to check that the orthogonal complement contains an ample class $A = a_1 H_1 + a_2 H_2 + a_3 H_3$ given in terms of positive integers a_1, a_2, a_3 , i.e.,

$$0 = A \cdot e = a_1 d_2 + a_2 d_1 + a_3 (d_1 + d_2 - 2k). \quad (5.25)$$

As the sum of the first two terms are always positive, this orthogonality condition can only be met if

$$d_1 + d_2 < 2k \Leftrightarrow (d_1 + d_2)^2 < 4k^2 \Leftrightarrow (d_2 - d_1)^2 < 4, \quad (5.26)$$

where we used the relations (5.24), which furthermore implies that $d_2 = d_1 + 1$ and hence $k = 1, d_1 = 0, d_2 = 1$, corresponding to the vector

$$e = -H_1 + H_3. \quad (5.27)$$

For this vector e the ample class $A = H_1 + H_2 + H_3$ is indeed orthogonal.

Therefore, for a left building block (Z_L, S_L) of the Fano threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a rank one intersection lattice R generated by a vector e with $e^2 = -4$ — up to trivial relabelling of the Kähler cone generators — the vector (5.27) is the only possibility. Its orthogonal complement W_L is then generated by

$$W_L = \mathbb{Z} w_1 + \mathbb{Z} w_2 \quad \text{with} \quad w_1 = H_1 + H_3, \quad w_2 = H_2, \quad (5.28)$$

such that the Picard lattice N_L in terms of (w_1, w_2, e) reads

$$N_L = \mathbb{Z} w_1 + \mathbb{Z} w_2 + \mathbb{Z} e + \frac{1}{2} \mathbb{Z} (w_1 + e). \quad (5.29)$$

So as to orthogonally glue this left Picard lattice N_L along e with a Picard lattice N_R of a right rank two Fano building block (Z_R, S_R) , we find in the Crowley–Nordström classification [56] that the rank two Fano threefolds with Mori–Mukai reference numbers MM6₂, MM12₂, MM21₂, and MM32₂ give rise to compatible intersection lattices. For convenience, these particular building blocks together with some geometric data are summarized in Table 5.2, and we readily see that for all these examples the rank two Picard lattices N_R are generated in the orthogonal basis (A, e) by

$$N_R = \mathbb{Z} A + \mathbb{Z} e + \frac{1}{2} \mathbb{Z} (A + e), \quad (5.30)$$

with $e^2 = -4$ and the length square of the ample class A as listed in Table 5.2.

Next we can construct the orthogonal pushout $W = N_L \perp_e N_R$, which in the basis (w_1, w_2, e, A) takes the form

$$W = \mathbb{Z} w_1 + \mathbb{Z} w_2 + \mathbb{Z} e + \mathbb{Z} A + \frac{1}{2} \mathbb{Z} (w_1 + e) + \frac{1}{2} \mathbb{Z} (A + e). \quad (5.31)$$

No.	$-K^3$	κ	Δ^κ	A	e	A^2	e^2	$b_3(Z)$
MM6 ₂	12	$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$	-12	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	12	-4	32
MM12 ₂	20	$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$	-20	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	20	-4	28
MM21 ₂	28	$\begin{pmatrix} 6 & 8 \\ 8 & 6 \end{pmatrix}$	-28	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	28	-4	30
MM32 ₂	48	$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$	-12	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	12	-4	50

Table 5.2: The rows of the table list the data of all rank two Fano threefolds admitting an intersection lattice R generated by a vector of length square -4 . The columns show the reference number in the Mori–Mukai classification, the triple intersection of the anti-canonical divisor $-K$, the intersection matrix κ of the Picard lattice of the anti-canonical K3 surface, the discriminant Δ^κ of the intersection matrix κ , the chosen ample class A in the basis of the intersection matrix κ , the orthogonal complement e to the ample class A , the length squares of the classes A and e , and the three-form Betti number $b_3(Z)$ of the associated building block Z .

This orthogonal pushout is well-defined, as the potentially non-integral intersection pairing $\langle \frac{1}{2}(w_1 + e), \frac{1}{2}(A + e) \rangle_W = -1$ is indeed integral. The intersection pairing κ_W of the pushout W becomes for the integral generators $(\frac{1}{2}(w_1 + e), w_2, \frac{1}{2}(A + e), e)$

$$\kappa_W = \begin{pmatrix} 0 & 2 & -1 & -2 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & \frac{1}{4}A^2 - 1 & -2 \\ -2 & 0 & -2 & -4 \end{pmatrix}, \quad \det \kappa_W = 4A^2, \quad (5.32)$$

where — according to Table 5.2 — the entry $\frac{1}{4}A^2 - 1$ is integral and even. As a result we obtain for the orthogonal pushouts W of the rank three Fano threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with reference number MM27₃ and the rank two Fano threefolds listed in Table 5.2 the twisted connected sum G_2 -manifold $Y_{\dots}^{\text{MM27}_3}$

$$\begin{aligned} W_{\text{MM6}_2}^{\text{MM27}_3} &= N_{\text{MM27}_3} \perp_e N_{\text{MM6}_2} : & b_2(Y_{\text{MM6}_2}^{\text{MM27}_3}) &= 1, & b_3(Y_{\text{MM6}_2}^{\text{MM27}_3}) &= 104, \\ W_{\text{MM12}_2}^{\text{MM27}_3} &= N_{\text{MM27}_3} \perp_e N_{\text{MM12}_2} : & b_2(Y_{\text{MM12}_2}^{\text{MM27}_3}) &= 1, & b_3(Y_{\text{MM12}_2}^{\text{MM27}_3}) &= 100, \\ W_{\text{MM21}_2}^{\text{MM27}_3} &= N_{\text{MM27}_3} \perp_e N_{\text{MM21}_2} : & b_2(Y_{\text{MM21}_2}^{\text{MM27}_3}) &= 1, & b_3(Y_{\text{MM21}_2}^{\text{MM27}_3}) &= 102, \\ W_{\text{MM32}_2}^{\text{MM27}_3} &= N_{\text{MM27}_3} \perp_e N_{\text{MM32}_2} : & b_2(Y_{\text{MM32}_2}^{\text{MM27}_3}) &= 1, & b_3(Y_{\text{MM32}_2}^{\text{MM27}_3}) &= 122. \end{aligned} \quad (5.33)$$

Analogously, we can construct twisted connected sum G_2 -manifolds via orthogonal gluing along rank one intersection lattices for semi-Fano threefolds with higher rank Picard lattices. In Table 5.3 we collect all (resolved) toric terminal Fano threefolds

of Picard rank three and four that allow for a rank one intersection lattice generated by a vector e of length square $e^2 = -4$. The geometries of these threefolds are again specified by their reference number $\text{MM}\#_\rho$ and/or $\text{K}\#$ as arising in the Mori–Mukai and Kasprzyk classifications [62–64]. The resulting twisted connected sum G_2 -manifolds $Y_{\text{---}}$ obtained from orthogonal gluing along the rank one intersection lattice R all have the two-form Betti number $b_2(Y_{\text{---}}) = 1$ and their three-form Betti numbers $b_3(Y_{\text{---}})$ are listed in Table 5.4. These Betti numbers are easily calculated with relations (5.11).

No.	rk N	$-K^3$	κ	e	e^2	$b_3(Z)$
MM27 ₃ , K62 (Fano)	3	48	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	-4	50
MM25 ₃ , K68 (Fano)	3	44	$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	-4	46
MM31 ₃ , K105 (Fano)	3	52	$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	-4	54
K124 (semi-Fano)	3	48	$\begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	-4	50
MM12 ₄ , K218 (Fano)	4	46	$\begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$	-4	48
MM10 ₄ , K266 (Fano)	4	42	$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$	-4	44
K221 (semi-Fano)	4	38	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$	-4	40
K232 (semi-Fano)	4	40	$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$	-4	42
K233 (semi-Fano)	4	38	$\begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	-4	40
K247 (semi-Fano)	4	44	$\begin{pmatrix} 4 & 3 & 3 & 2 \\ 3 & 0 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$	-4	46
K257 (semi-Fano)	4	46	$\begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & -2 & 1 \\ 3 & 3 & 1 & 6 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	-4	48

Table 5.3: The rows of the table list the data of all rank three and four (resolved) toric terminal Fano threefolds for an intersection lattice R generated by a vector of length square -4 . The columns show the reference number in the Mori–Mukai [62] and/or Kasprzyk [64] classification, the rank of the Picard lattice —note that in the semi-Fano cases this rank as reported in [64] is smaller as it refers to the singular variety—, the triple intersection of the anti-canonical divisor $-K$, the intersection matrix κ of the Picard lattice of the anti-canonical K3 surface, the generator e of the lattice R and its length square, and the three-form Betti number $b_3(Z)$ of the associated building block Z .

$b_3(Y_{\dots})$	MM27 ₃	MM25 ₃	MM31 ₃	K124	MM12 ₄	MM10 ₄	K221	K232	K233	K247	K257
MM27 ₃	122	118	126	122	120	116	112	114	112	118	120
MM25 ₃	118	114	122	118	116	112	108	110	108	114	116
MM31 ₃	126	122	130	126	124	120	116	118	116	122	124
K124	122	118	126	122	120	116	112	114	112	118	120
MM12 ₄	120	116	124	120	118	114	110	112	110	116	118
MM10 ₄	116	112	120	116	114	110	106	108	106	112	114
K221	112	108	116	112	110	106	102	104	102	108	110
K232	114	110	118	114	112	108	104	106	104	110	112
K233	112	108	116	112	110	106	102	104	102	108	110
K247	118	114	122	118	116	112	108	110	108	114	116
K257	120	116	124	120	118	114	110	112	110	116	118

Table 5.4: This table shows the three-form Betti numbers $b_3(Y_{\dots})$ of the twisted connected sum G_2 -manifolds Y_{\dots} arising from the orthogonal pushout $N_{\dots} \perp_e N_{\dots}$ along the rank one intersection lattice with $e^2 = -4$ from all pairs of building blocks collected in Table 5.3. By construction of gluing along a rank one intersection lattice all these examples have the two-form Betti numbers $b_2(Y_{\dots}) = 1$. The reference numbers MM# _{ρ} or K# for the rows and columns label the building blocks, and the lines in the table divides between the examples with rank three and rank four Picard lattices.

Orthogonal gluing along rank two intersection lattice: A systematic analysis of orthogonal pushouts for higher rank intersection lattices R is beyond the scope of this paper. Instead, we present a particular example with a rank two intersection lattice R with two orthogonal generators e_1 and e_2 both of length square -4 . Certainly we do not expect that orthogonality and the maximal negative value for the length squares are necessary conditions to find a higher rank example. However, imposing these two conditions certainly simplifies the construction of a matching pair.

Our example is based upon gluing a pair of building blocks $(Z_{L/R}, S_{L/R})$ both obtained from the rank five Fano threefold $P_{L/R} = \mathbb{P}^1 \times dP_3$, where dP_3 denotes the del Pezzo surface of degree six, which is the blow-up of \mathbb{P}^2 along three non-collinear points p_1, p_2, p_3 . This rank five Fano threefold has the Mori–Mukai reference number MM3₅ and — as it is toric — the Kasprzyk reference number K324.

First, we collect some basic properties of the del Pezzo surface dP_3 . Let E_1, E_2, E_3 be the three exceptional divisors from the blow-ups at the points p_1, p_2, p_3 , and let H be the proper transform of the hyperplane class of \mathbb{P}^2 . These divisors span the Picard lattice of dP_3 and their intersection numbers read

$$\langle E_i, E_j \rangle_{dP_3} = -\delta_{ij} , \quad \langle H, H \rangle_{dP_3} = 1 , \quad \langle H, E_i \rangle_{dP_3} = 0 . \quad (5.34)$$

The ample anti-canonical divisor reads $-K_{dP_3} = 3H - E_1 - E_2 - E_3$.

Let us further define the two divisors

$$e_1 = E_1 + E_2 + E_3 - H , \quad e_2 = E_1 - E_2 , \quad (5.35)$$

which are both differences of rational curves on dP_3 . The most important point, however, is that the defined divisors e_1, e_2 of length square -2 are both mutual orthogonal and orthogonal to the class $-K_{dP_3}$ in the Kähler cone $\mathcal{K}(dP_3)$, i.e.,

$$\langle e_1, e_2 \rangle_{dP_3} = \langle e_1, K_{dP_3} \rangle_{dP_3} = \langle e_2, K_{dP_3} \rangle_{dP_3} = 0, \quad \langle e_1, e_1 \rangle_{dP_3} = \langle e_2, e_2 \rangle_{dP_3} = -2. \quad (5.36)$$

Now we return to the rank five Fano threefold $\mathbb{P}^1 \times dP_3$. With the hyperplane divisor h of \mathbb{P}^1 and the described divisors of dP_3 the anti-canonical divisor becomes

$$-K_{\mathbb{P}^1 \times dP_3} = 2h - K_{dP_3} = 2h + 3H - E_1 - E_2 - E_3, \quad (5.37)$$

Furthermore, the Picard lattice N of the polarized K3 surface S on $\mathbb{P}^1 \times dP_3$ is generated by the divisors h, H, E_1, E_2, E_3 together with the intersection pairing

$$\langle h, h \rangle_N = 0, \quad \langle h, D \rangle_N = -\langle K_{dP_3}, D \rangle_{dP_3}, \quad \langle D, F \rangle_N = 2\langle D, F \rangle_{dP_3}. \quad (5.38)$$

Here D and F are some divisors on dP_3 .

For the orthogonal pushout we generate the rank two lattice R with the two del Pezzo divisors e_1 and e_2 as

$$R = \mathbb{Z}e_1 + \mathbb{Z}e_2, \quad \langle e_i, e_j \rangle_N = -4\delta_{ij}, \quad (5.39)$$

where eqs. (5.36) and (5.38) determines the intersection pairing on R . Moreover, the orthogonal complement W of R becomes

$$W = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3 \quad \text{with} \quad w_1 = h - K_{dP_3}, \quad w_2 = H - E_3, \quad w_3 = h, \quad (5.40)$$

where in particular the ample generator w_1 is in the Kähler cone $\mathcal{K}(\mathbb{P}^1 \times dP_3)$. As a result for the rank five Picard lattice N of the polarized K3 surface S in $\mathbb{P}^1 \times dP_3$ we arrive with $(w_1, w_2, w_3, e_1, e_2)$ at

$$N = (\mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3) + (\mathbb{Z}e_1 + \mathbb{Z}e_2) + \frac{1}{2}(\mathbb{Z}(w_1 + e_1) + \mathbb{Z}(w_1 + w_2 + e_2)). \quad (5.41)$$

Now taking the decomposition (5.41) of the Picard lattice for both the left and the right Picard lattice, i.e., $N_L = N_R = N$, we consider the orthogonal pushout $W = N_L \perp_R N_R$, which in the basis $(w_1^L, w_2^L, w_3^L, w_1^R, w_2^R, w_3^R, e_1, e_2)$ takes the form

$$\begin{aligned} W = & (\mathbb{Z}w_1^L + \mathbb{Z}w_2^L + \mathbb{Z}w_3^L) + (\mathbb{Z}w_1^R + \mathbb{Z}w_2^R + \mathbb{Z}w_3^R) \\ & + (\mathbb{Z}e_1 + \mathbb{Z}e_2) + \frac{1}{2}(\mathbb{Z}(w_1^L + e_1) + \mathbb{Z}(w_1^R + e_1)) \\ & + \frac{1}{2}(\mathbb{Z}(w_1^L + w_2^L + e_2) + \mathbb{Z}(w_1^R + w_2^R + e_2)). \end{aligned} \quad (5.42)$$

This orthogonal pushout is well-defined because the potentially non-integral intersections $\langle \frac{1}{2}(w_1^L + e_1), \frac{1}{2}(w_1^R + e_1) \rangle = \langle \frac{1}{2}(w_1^L + w_2^L + e_2), \frac{1}{2}(w_1^R + w_2^R + e_2) \rangle_W = -1$ and $\langle \frac{1}{2}(w_1^L + e_1), \frac{1}{2}(w_1^R + w_2^R + e_2) \rangle_W = \langle \frac{1}{2}(w_1^R + e_1), \frac{1}{2}(w_1^L + w_2^L + e_2) \rangle_W = 0$ are integral.

As a result we obtain from this orthogonal pushout along the rank two lattice R the twisted connected G_2 -manifold $Y_{\text{MM35}}^{\text{MM35}}$ with the Betti numbers

$$W_{\text{MM35}}^{\text{MM35}} = N_{\text{MM35}} \perp_R N_{\text{MM35}} : \quad b_2(Y_{\text{MM35}}^{\text{MM35}}) = 2, \quad b_3(Y_{\text{MM35}}^{\text{MM35}}) = 97. \quad (5.43)$$

Here we use that $b_3(Z_{L/R}) = \langle K_{\mathbb{P}^1 \times dP_3}, K_{\mathbb{P}^1 \times dP_3} \rangle_N + 2 = 6 \langle K_{dP_3}, K_{dP_3} \rangle_{dP_3} + 2 = 38$ because $b_3(\mathbb{P}^1 \times dP_3) = 0$, c.f., ref. [62].

6 $\mathcal{N} = 2$ gauge sectors on twisted connected sums

Compared to the Abelian $\mathcal{N} = 4$ gauge theory sectors studied in the previous section, the structure of the $\mathcal{N} = 2$ gauge theory sectors turns out to be much richer. The building blocks $(Z_{L/R}, S_{L/R})$ of the twisted connected G_2 -manifolds admit enhancement to $\mathcal{N} = 2$ non-Abelian gauge theory sectors with an interesting branch structure that is geometrically accessible in terms of extremal transitions in the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$.

In order to see what kind features we can expect by degenerating the building blocks $(Z_{L/R}, S_{L/R})$, we recall from ref. [33] that there is a simple hierarchy of real codimension four, six and seven singularities in G_2 -manifolds, which respectively lead to non-Abelian gauge groups, non-trivial matter representations, and chirality of the charged $\mathcal{N} = 1$ matter spectrum. While our setup admits non-Abelian gauge groups with non-trivial matter representations, we should not expect singularities inducing chirality, as the trivial S^1 fibration in the non-compact seven-manifolds $Y_{L/R}$ prevents the appearance of codimension seven singularities.

The picture proposed in ref. [33] uses the heterotic/M-theory duality [66], the Strominger–Yau–Zaslow fibration of Calabi–Yau manifolds [68], and the fact that G_2 -manifolds can locally be constructed as degenerating S^1 fibrations over Calabi–Yau threefolds [36, 69], where the S^1 can be identified in a hyper Kähler quotient construction starting in eight dimensions [37, 33]. Namely, consider the heterotic string compactification on the Calabi–Yau threefold W . We further assume that the threefold W admits a geometric mirror threefold, such that it has a Strominger–Yau–Zaslow Lagrangian T^3 fibration over a (real) three-dimensional Lagrangian cycle Q .²⁸ In the best known examples — such as hypersurfaces in toric varieties — it has the topology of a three-sphere.²⁹ In the limit where the volume of the base Q is large

²⁸Further suggestions on the role of mirror symmetry in the context of G_2 -manifolds have been proposed in refs. [70, 71].

²⁹It is possible to obtain a Lens space for the Lagrangian base Q , for instance by dividing out a freely acting finite group on a suitable Calabi–Yau threefold, see e.g., ref. [72].

compared to the volume of the Lagrangian fibers T^3 , the essential idea is now to adiabatically extend the duality between the heterotic string on T^3 and M-theory on $K3$ over the entire base Q . The M-theory geometries defined in this way realize the same fibration structure (4.6) as appearing in the twisted connected sum G_2 -manifolds. Whenever the heterotic string has a non-Abelian ADE type gauge group G , the dual K3 fibers develop the corresponding ADE singularity extending over the entire real three-dimensional base Q . The proposed construction can be viewed as an $\mathcal{N} = 1$ version of the $\mathcal{N} = 2$ heterotic/type II duality between the heterotic string on $K3 \times T^2$ and type IIA string on the dual K3-fibered Calabi–Yau threefolds, as proposed in refs. [73, 74]. In the context of twisted connected sum G_2 -manifolds a possibility to arrive at non-Abelian ADE type gauge theories has been contemplated in ref. [35].

While the K3 fibration described in ref. [33] complies with the K3 fibration (4.6) of the twisted connected sum construction, we should stress that the non-Abelian gauge theory enhancement encountered in this work arises from singularities along a three-cycle $S^1 \times \mathcal{C}$, where the curve \mathcal{C} of genus g resides in K3-fibers along a circle S^1 in the base Q . Thus, compared to ref. [33] the non-Abelian gauge group emerges still from a real codimension four singularity, however, along different types of three-cycles. In the Kovalev limit the three-cycle $S^1 \times \mathcal{C}$ resides in one of the seven-manifold $Y_{L/R} = X_{L/R} \times S^1_{L/R}$ such that the curve \mathcal{C} realizes an ADE singularity in one of the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$. Therefore, the non-Abelian gauge theory enhancement discussed here directly relates to non-Abelian gauge groups from curves of ADE singularities in Calabi–Yau threefolds in the context of type IIA strings [40, 41]. Specifically, in this setting an ADE singularity along a curve \mathcal{C} yields a four-dimensional $\mathcal{N} = 2$ gauge theory with the associated gauge group G together with g hypermultiplets in the adjoint representation. More general matter representations occur from points along \mathcal{C} where the ADE singularity further enhances, i.e., along real codimension six singularities. For instance, at the intersection point of two curves \mathcal{C} and \mathcal{C}' of ADE singularities we encounter matter in the bi-fundamental representation of the two associated gauge groups G and G' [43]. We find in the following that the described $\mathcal{N} = 2$ gauge theory spectra can indeed be realized within the $\mathcal{N} = 2$ gauge theory sectors of the building blocks $(Z_{L/R}, S_{L/R})$. Remarkably, even the phase structure of the four-dimensional $\mathcal{N} = 2$ gauge theory sectors — connecting topological distinct Calabi–Yau threefolds via extremal transitions — carries over to the four-dimensional $\mathcal{N} = 1$ M-theory compactifications on twisted connected sum G_2 -manifolds, where now the gauge theory branches relate topologically distinct G_2 -manifolds.

The picture presented in refs. [32, 66] finally explains chirality of non-Abelian matter as a local effect that occurs in codimension seven. Since the twisted connected sum breaks supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ non-locally via the twisted gluing recipe, the last step — namely to construct chiral charged matter — is more subtle

to achieve. We discuss some ideas in the conclusions.

6.1 Phases of $\mathcal{N} = 2$ Abelian gauge theory sectors

Let us now focus on twisted connected sum G_2 -manifolds with non-trivial $\mathcal{N} = 2$ gauge theory sectors in the Kovalev limit. According to Table 4.1 this amounts to constructing building blocks with $(Z_{L/R}, S_{L/R})$ with non-trivial kernels $k_{L/R}$ as defined below eq. (4.1). This can be achieved with the proposal by Kovalev and Lee [75], generalizing the construction of asymptotically cylindrical Calabi–Yau threefolds outlined in Section 5.1. In a particular example the possibility to realize $\mathcal{N} = 2$ Abelian gauge theory enhancement appeared in ref. [25].

For the semi-Fano threefold P we pick again two global sections s_0 and s_1 of the anti-canonical divisor $-K_P$. However, instead of choosing a generic section s_0 , we assume that the global section s_0 factors into a product

$$s_0 = s_{0,1} \cdots s_{0,n} , \quad (6.1)$$

such that $s_{0,i}$ are global sections of line bundles \mathcal{L}_i with $-K_P = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$. As a consequence the curve $\mathcal{C}_{\text{sing}} = \{s_0 = 0\} \cap \{s_1 = 0\}$ becomes reducible and decomposes into

$$\mathcal{C}_{\text{sing}} = \sum_{i=1}^n \mathcal{C}_i , \quad \mathcal{C}_i = \{s_{0,i} = 0\} \cap \{s_1 = 0\} , \quad (6.2)$$

where we assume that the individual curves \mathcal{C}_i are smooth and reduced. Following Kovalev and Lee [75], we construct the building block (Z^\sharp, S) associated to P by the sequence of blow-ups $\pi_{\{\mathcal{C}_1, \dots, \mathcal{C}_n\}} : Z^\sharp \rightarrow P$ along the individual curves \mathcal{C}_i according to

$$Z^\sharp = \text{Bl}_{\{\mathcal{C}_1, \dots, \mathcal{C}_n\}} P = \text{Bl}_{\mathcal{C}_n} \text{Bl}_{\mathcal{C}_{n-1}} \cdots \text{Bl}_{\mathcal{C}_1} P . \quad (6.3)$$

Since the curves \mathcal{C}_i and the semi-Fano threefold P are smooth, the blow-up Z^\sharp is smooth as well. As before, the K3 surface S arises as the proper transform of a smooth anti-canonical divisor $S^\sharp = \{\alpha_0 s_0 + \alpha_1 s_1 = 0\} \subset P$ for some $[\alpha_0 : \alpha_1] \in \mathbb{P}^1$. By blowing up a semi-Fano threefold P the resulting dimension of the kernel k — defined below eq. (4.1) — is then given by [27]

$$\dim k = n - 1 . \quad (6.4)$$

Furthermore, the three-form Betti number $b_3(Z^\sharp)$ of the blown-up threefold Z^\sharp becomes

$$b_3(Z^\sharp) = b_3(P) + 2 \sum_{i=1}^n g(\mathcal{C}_i) , \quad (6.5)$$

in terms of the three-form Betti number $b_3(P)$ of the semi-Fano threefold P and the genera $g(\mathcal{C}_i)$ of the smooth curve components \mathcal{C}_i . As all these curves \mathcal{C}_i lie in the K3

Multiplicity	$\mathcal{N} = 2$ multiplets		$\mathcal{N} = 1$ multiplets	
	$U(1)^{n-1}$ charges	multiplet	$U(1)^{n-1}$ charges	multiplet
$n - 1$	$(0, 0, \dots, 0)$	vector	$(0, \dots, 0)$ $(0, \dots, 0)$	vector chiral
χ_{ij} $1 \leq i < j < n$	$(0, \dots, +1_i, \dots, +1_j, \dots, 0)$	hyper	$(0, \dots, +1_i, \dots, +1_j, \dots, 0)$ $(0, \dots, -1_i, \dots, -1_j, \dots, 0)$	chiral chiral
χ_{in} $1 \leq i < n$	$(0, \dots, +1_i, \dots, 0)$	hyper	$(0, \dots, +1_i, \dots, 0)$ $(0, \dots, -1_i, \dots, 0)$	chiral chiral

Table 6.1: The table shows the spectrum of the Abelian $\mathcal{N} = 2$ gauge theory sector arising from the conifold singularities in the building block (Z_{sing}, S) . Listed are the four-dimensional $\mathcal{N} = 2$ multiplets and their decomposition into the four-dimensional $\mathcal{N} = 1$ multiplets together with their multiplicities χ_{ij} . The subscripts of the entries of the $U(1)$ charges indicate their position in the charge vector.

fiber S , the genus $g(\mathcal{C}_i)$ is readily computed by the adjunction formula

$$g(\mathcal{C}_i) = \frac{1}{2} \mathcal{C}_i \cdot \mathcal{C}_i + 1, \quad (6.6)$$

with the self-intersections $\mathcal{C}_i \cdot \mathcal{C}_i$ in S .

Corti et. al. show in refs. [24, 27] that with such a building block (Z^\sharp, S^\sharp) the orthogonal gluing recipe of Section 5.2 can still be carried out in the same way. In particular, the lift from a matching pair of K3 surfaces $S_{L/R}$ to their asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$ still exists if such generalized building blocks are involved in the orthogonal gluing procedure.

Thus, we observe that from a single semi-Fano threefold P often several building blocks can be constructed depending on the properties of the curve $\mathcal{C} = \{s_0 = 0\} \cap \{s_1 = 0\}$. Namely, for smooth irreducible curves \mathcal{C}^b we obtain a smooth building block (Z^b, S^b) with vanishing kernel k , whereas for a reducible curves $\mathcal{C}_{\text{sing}}$ with smooth components \mathcal{C}_i we arrive with the sequence of blow-ups (6.3) at a smooth building block (Z^\sharp, S^\sharp) with non-vanishing kernel k . The former building block does not contribute any vector multiplets for a $\mathcal{N} = 2$ gauge theory sector, while the latter building block contributes Abelian vector multiplets to the $\mathcal{N} = 2$ gauge theory sector. In the sequel we argue that these different possibilities realize distinct branches of the $\mathcal{N} = 2$ gauge theory sectors.

To arrive at this gauge theory interpretation, let us consider a semi-Fano threefold P with a curve $\mathcal{C}_{\text{sing}}$ of the reducible type (6.2) of the factorized global anti-canonical section (6.1). Performing a blow-up along this reducible curve yields the fibration

$\pi : Z_{\text{sing}} \rightarrow \mathbb{P}^1$ with

$$Z_{\text{sing}} = \text{Bl}_{\mathcal{C}_{\text{sing}}} P = \left\{ (x, z) \in P \times \mathbb{P}^1 \mid z_0 s_0^{(1)} \cdots s_0^{(n)} + z_1 s_1 = 0 \right\} . \quad (6.7)$$

In the vicinity of the fiber $\pi^{-1}([1, 0])$ the threefold Z_{sing} becomes singular because in the patch of the affine coordinate $t = \frac{z_1}{z_0}$ we get

$$s_{0,1} \cdots s_{0,n} + t s_1 = 0 . \quad (6.8)$$

Thus — assuming transverse intersections among the smooth curves \mathcal{C}_i — there are conifold singularities at the discrete intersection loci $\mathcal{I}_{ij} = \{t = 0\} \cap \{s_1 = 0\} \cap \{s_{0,i} = 0\} \cap \{s_{0,j} = 0\}$ for $1 \leq i \leq j \leq n$ with $\chi_{ij} = |\mathcal{I}_{ij}|$ intersection points. These numbers are given by

$$\chi_{ij} = \mathcal{C}_i \cdot \mathcal{C}_j , \quad (6.9)$$

in terms of the intersection numbers of the reduced curve \mathcal{C}_i and \mathcal{C}_j within the K3 surface S . This singularity structure prevails in the asymptotically cylindrical Calabi–Yau threefold $X_{\text{sing}} = Z_{\text{sing}} \setminus S$ as the asymptotic fiber $S = \pi^{-1}([\alpha_0, \alpha_1])$ (for $\alpha_1 \neq 0$) is disjoint from the singular fiber $\pi^{-1}([1, 0])$.

In the vicinity of the singular fiber $\pi^{-1}([1, 0]) \subset X_{\text{sing}}$, we interpret the dimensional reduction of M-theory on the local seven-dimensional singular space $S^1 \times X_{\text{sing}}$ as the dimensional reduction of IIA string theory on the asymptotically cylindrical Calabi–Yau threefold X_{sing} , where the S^1 factor corresponds to the M-theory circle of type IIA string theory. In this IIA picture refs. [39, 38] establish that the conifold singularities (6.8) yield an Abelian $\mathcal{N} = 2$ gauge theory with charged matter multiplets.³⁰ Namely, to each curve \mathcal{C}_i we assign an Abelian group factor $U(1)_i$ such that the total Abelian gauge group of rank $n - 1$ becomes

$$U(1)^{n-1} \simeq \frac{U(1)_1 \times \cdots \times U(1)_n}{U(1)_{\text{Diag}}} , \quad (6.10)$$

where $U(1)_{\text{Diag}}$ is the diagonal subgroup of $U(1)_1 \times \cdots \times U(1)_n$. Thus, in the low-energy effective theory we obtain in addition to $(n - 1)$ four-dimensional $\mathcal{N} = 2$ $U(1)$ vector multiplets, which decomposes into $(n - 1)$ four-dimensional $\mathcal{N} = 1$ $U(1)$ vector multiplets and $n - 1$ four-dimensional $\mathcal{N} = 1$ neutral chiral multiplets. Furthermore, to each intersection point in \mathcal{I}_{ij} one assigns a four-dimensional $\mathcal{N} = 2$ hypermultiplet of charge $(+1, +1)$ with respect to the $U(1)_i \times U(1)_j$ group factor. Then each of these $\mathcal{N} = 2$ hypermultiplet of charge $(+1, +1)$ decomposes into two four-dimensional $\mathcal{N} = 1$ chiral multiplets of charge $(+1, +1)$ and $(-1, -1)$, respectively. We summarize the resulting spectrum in Table 6.1.

³⁰Starting from the Fano threefold \mathbb{P}^3 , it has also been proposed in ref. [25] that the singular building block (Z_{sing}, S) with conifold singularities realizes an Abelian gauge theory with charged matter.

Alternatively, we can dimensionally reduce M-theory on the local Calabi–Yau fourfold $T^2 \times X_{\text{sing}}$ to three space-time dimensions. Then the conifold points in X_{sing} become genus one curves of conifold singularities. This analysis has been carried out in ref. [76], and one arrives at a three-dimensional $\mathcal{N} = 4$ gauge theory sectors, which agree with the four-dimensional $\mathcal{N} = 2$ spectrum in Table 6.1 upon further dimensional reduction on a circle S^1 .³¹ This further justifies that the local dimensional reduction of type IIA theory on X_{sing} correctly describes the gauge theory of M-theory on $S^1 \times X_{\text{sing}}$ without requiring that the S^1 factor realizes the M-theory circle for the dual type IIA description.

The described four-dimensional $\mathcal{N} = 2$ Abelian gauge theory now predicts a Higgs branch H^b and a Coulomb branch C^\sharp . On the one hand, generic non-vanishing expectation values of the charged hypermultiplets break the $U(1)^{n-1}$ gauge theory entirely and parametrize the Higgs branch H^b of the gauge theory. As a consequence $(n-1)$ charged $\mathcal{N} = 2$ hypermultiplets play the role of $\mathcal{N} = 2$ Goldstone multiplets that combine with the $(n-1)$ short massless $\mathcal{N} = 2$ vector multiplets to $(n-1)$ long massive $\mathcal{N} = 2$ vector multiplets. As a result — according to the spectrum in Table 6.1 — we arrive at the Higgs branch H^b of complex dimension h^b [38]

$$h^b = \dim_{\mathbb{C}} H^b = 2 \left(\sum_{1 \leq i < j \leq n} \chi_{ij} \right) - 2(n-1) . \quad (6.11)$$

Here the factor two takes into account that each hypermultiplet contains two complex scalar fields. This complex dimension readily describes the Higgs branch as parametrized by the expectation values of the corresponding charged $\mathcal{N} = 1$ chiral multiplets. On the other hand the expectation values of the neutral complex scalar fields in the $\mathcal{N} = 2$ vector multiplets furnish the coordinates on the Coulomb branch C^\sharp such that its complex dimension c^\sharp reads

$$c^\sharp = \dim_{\mathbb{C}} C^\sharp = n-1 . \quad (6.12)$$

In the $\mathcal{N} = 1$ language the Coulomb branch moduli space is parametrized by the expectation value of neutral $\mathcal{N} = 1$ chiral multiplets.

In the geometry, the Higgs branch H^b arises from deforming the conifold singularities in X_{sing} to the deformed asymptotically cylindrical Calabi–Yau threefold X^b [38]. On the level of the semi-Fano threefold P this amounts to deforming the reducible curve $\mathcal{C}_{\text{sing}}$ in eq. (6.2) to the smooth reduced curve \mathcal{C}^b such that the building block (Z_{sing}, S) deforms to the building block (Z^b, S^b) , which yields for the kernel k^b and the three-form Betti number $b_3(Z^b)$ according to eqs. (6.4) and (6.5)

$$\dim k^b = 0 , \quad b_3(Z^b) = b_3(P) + \mathcal{C}^b \cdot \mathcal{C}^b + 2 . \quad (6.13)$$

³¹Note that eight supercharges correspond to $\mathcal{N} = 4$ supersymmetry in three and to $\mathcal{N} = 2$ supersymmetry in four dimensions.

Furthermore, the resolution of the conifold singularities in X_{sing} geometrically yields the Coulomb branch C^\sharp of the gauge theory [38], which again on the level of the semi-Fano threefold P realizes the sequential blow-ups (6.3) along the components \mathcal{C}_i of $\mathcal{C}_{\text{sing}}$ to the building block (Z^\sharp, S^\sharp) .³² With eqs. (6.4) and (6.5) the dimension of its kernel k^\sharp and the Betti number $b_3(Z^\sharp)$ becomes

$$\dim k^\sharp = n - 1 , \quad b_3(Z^\sharp) = b_3(P) + 2n + \sum_{i=1}^n \mathcal{C}_i \cdot \mathcal{C}_i . \quad (6.14)$$

Let us now consider two twisted connected sum G_2 -manifolds Y^\flat and Y^\sharp respectively constructed via orthogonal gluing of the left building blocks (Z^\flat, S^\flat) and (Z^\sharp, S^\sharp) with another right building block (Z_R, S_R) . Then we can use eq. (5.11) to deduce from eqs. (6.13) and (6.14) the Betti numbers

$$b_2(Y^\flat) = \delta_R^{(2)} , \quad b_3(Y^\flat) = b_3(P) + \mathcal{C}^\flat \cdot \mathcal{C}^\flat + 25 + \delta_R^{(3)} , \quad (6.15)$$

and

$$b_2(Y^\sharp) = (n - 1) + \delta_R^{(2)} , \quad b_3(Y^\sharp) = b_3(P) + \left(\sum_{i=1}^n \mathcal{C}_i \cdot \mathcal{C}_i \right) + 3n + 22 + \delta_R^{(3)} , \quad (6.16)$$

with the contributions from the right building block (Z_R, S_R)

$$\delta_R^{(2)} = \dim k_R + \text{rk } R , \quad \delta_R^{(3)} = b_3(Z_R) + \dim k_R - \text{rk } R . \quad (6.17)$$

The relations allow us to further define what we call the reduced Betti numbers b_ℓ^\flat and b_ℓ^\sharp , $\ell = 1, 2$, given by

$$b_\ell^\flat = b_\ell(Y^\flat) - \delta_R^{(\ell)} , \quad b_\ell^\sharp = b_\ell(Y^\sharp) - \delta_R^{(\ell)} , \quad \ell = 1, 2 . \quad (6.18)$$

Using the equivalence $\mathcal{C}^\flat \sim \mathcal{C}_1 + \dots + \mathcal{C}_n$ on the semi-Fano threefold P and the definition (6.9) of the multiplicities χ_{ij} , we finally arrive at

$$\begin{aligned} b_2(Y^\flat) &= b_2(Y^\sharp) - (n - 1) , \\ b_3(Y^\flat) &= b_3(Y^\sharp) + 2 \left(\sum_{1 \leq i < j \leq n} \chi_{ij} \right) - 3(n - 1) . \end{aligned} \quad (6.19)$$

The non-trivial result is now that the derived change in Betti numbers (6.19) between such twisted connected sum G_2 -manifolds is in perfect agreement with the

³²The sequential resolution (6.3) in the Coulomb branch depends on the order of the performed blow-ups. Changing the order geometrically realizes bi-rational transformations among the resulting asymptotically cylindrical Calabi–Yau threefolds X^\sharp , as discussed in ref. [25].

phase structure of the proposed $U(1)^{n-1}$ gauge theory. The change in the Betti number b_2 geometrically realizes the difference of massless four-dimensional $\mathcal{N} = 1$ vector multiplets, whereas the difference of four-dimensional $\mathcal{N} = 1$ chiral multiplets reflects the change of the Betti number b_3 . This is in agreement with the gauge theory expectation. Passing from the Coulomb branch C^\sharp to the Higgs branch H^\flat via the Higgs mechanism reduces the vector bosons by the rank $(n-1)$ of the gauge group. Furthermore, the difference in the four-dimensional $\mathcal{N} = 1$ chiral multiplets agrees with the change in dimension of the moduli space of these gauge theory phases, i.e.,

$$b_3(Y^\flat) - b_3(Y^\sharp) = b_3^\flat - b_3^\sharp = h^\flat - c^\sharp . \quad (6.20)$$

In a similar fashion it is straightforward to establish the correspondence between the gauge theory and the geometry for mixed Higgs–Coulomb branches, where the gauge group $U(1)^{n-1}$ is broken to a subgroup $U(1)^{k-1}$ with $1 < k < n$. The geometries of such mixed phases are obtained by partially resolving and partially deforming the conifold singularities in the asymptotically cylindrical Calabi–Yau threefold X_{sing} . We illustrate the analysis of mixed Higgs–Coulomb branches with an explicit example in Section 6.3.

6.2 Phases of $\mathcal{N} = 2$ non-Abelian gauge theory sectors

Let us now turn to the enhancement to non-Abelian $\mathcal{N} = 2$ gauge theory sectors in the context of twisted connected G_2 -manifolds, indicated as a possibility in ref. [35]. Let us assume that the anti-canonical line bundle $-K_P$ of the semi-Fano threefold P factors as

$$-K_P = \tilde{\mathcal{L}}_1^{\otimes k_1} \otimes \dots \otimes \tilde{\mathcal{L}}_s^{\otimes k_s} \quad \text{with} \quad n = k_1 + \dots + k_s , \quad (6.21)$$

where $\tilde{\mathcal{L}}_i$ are line bundles with global sections $\tilde{s}_{0,i}$. Then the global section s_0 of $-K_P$ can further degenerate to $s_0 = \tilde{s}_{0,1}^{k_1} \dots \tilde{s}_{0,s}^{k_s}$ and the singular building block (6.7) reads

$$Z_{\text{sing}} = \{ (x, z) \in P \times \mathbb{P}^1 \mid z_0 \tilde{s}_{0,1}^{k_1} \dots \tilde{s}_{0,s}^{k_s} + z_1 s_1 = 0 \} , \quad (6.22)$$

with the singular equation in the affine coordinate $t = \frac{z_1}{z_0}$ given by

$$\tilde{s}_{0,1}^{k_1} \dots \tilde{s}_{0,s}^{k_s} + t s_1 = 0 . \quad (6.23)$$

As before we assume that all curves $\tilde{\mathcal{C}}_i = \{\tilde{s}_{0,i} = 0\} \cap \{s_1 = 0\}$ are smooth. In the vicinity of the singular fiber $\pi^{-1}([1, 0]) \subset Z_{\text{sing}}$ the singular building block (Z_{sing}, S) develops A_{k_i-1} -singularities along those curves $\tilde{\mathcal{C}}_i$ with $k_i > 1$.

To arrive at the gauge theory description we again analyze the local M-theory geometry on $S^1 \times X_{\text{sing}}$ in terms of its dual type IIA picture on the asymptotically cylindrical Calabi–Yau threefold X_{sing} . Refs. [40, 41] establish that type IIA string theory on Calabi–Yau threefolds with a genus g curve of A_{k-1} singularities develops a $\mathcal{N} = 2$ $SU(k)$ gauge theory with g four-dimensional $\mathcal{N} = 2$ hypermultiplets

Multiplicity	$\mathcal{N} = 2$ multiplets		$\mathcal{N} = 1$ multiplets	
	G reps.	multiplet	G reps.	multiplet
$s - 1$	1	$U(1)$ vector	1 1	$U(1)$ vector chiral
$i = 1, \dots, s$	$\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector	$\mathbf{adj}_{SU(k_i)}$ $\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector chiral
$g(\tilde{\mathcal{C}}_i)$ $1 \leq i \leq s$	$\mathbf{adj}_{SU(k_i)}$	hyper	$\mathbf{adj}_{SU(k_i)}$ $\mathbf{adj}_{SU(k_i)}$	chiral chiral
$\tilde{\chi}_{ij}$ $1 \leq i < j < s$	$(\mathbf{k}_i, \mathbf{k}_j)_{(+1_i, +1_j)}$	hyper	$(\mathbf{k}_i, \mathbf{k}_j)_{(+1_i, +1_j)}$ $(\bar{\mathbf{k}}_i, \bar{\mathbf{k}}_j)_{(-1_i, -1_j)}$	chiral chiral
$\tilde{\chi}_{in}$ $1 \leq i < n$	$(\mathbf{k}_i, \mathbf{k}_n)_{(+1_i)}$	hyper	$(\mathbf{k}_i, \mathbf{k}_n)_{(+1_i)}$ $(\bar{\mathbf{k}}_i, \bar{\mathbf{k}}_n)_{(-1_i)}$	chiral chiral

Table 6.2: The table shows the spectrum of the $\mathcal{N} = 2$ gauge theory sector with gauge group $G = SU(k_1) \times \dots \times SU(k_s) \times U(1)^{s-1}$ as arising from the non-Abelian building blocks (Z_{sing}, S) . It lists both the four-dimensional $\mathcal{N} = 2$ and the four-dimensional $\mathcal{N} = 1$ multiplet structure. The adjoint matter is determined by the genus $g(\tilde{\mathcal{C}}_i)$ of the curves $\tilde{\mathcal{C}}_i$, whereas the bi-fundamental matter from their intersection numbers $\tilde{\chi}_{ij}$ within the K3 surface S .

in the adjoint representation of $SU(N)$. Furthermore, according to ref. [43] each intersection point of two such curves of A_{k_1-1} and A_{k_2-1} singularities contributes a four-dimensional $\mathcal{N} = 2$ hypermultiplet in the bi-fundamental representation $(\mathbf{k}_1, \mathbf{k}_2)$ of $SU(k_1) \times SU(k_2)$.

Therefore — putting all these ingredients together and including the $U(1)$ gauge theory factors of the previously discussed Abelian gauge theory sectors — we propose for M-theory on the local singular seven space $S^1 \times X_{\text{sing}}$ the following non-Abelian gauge theory description. Firstly, the singularities along the curves $\tilde{\mathcal{C}}_i$ determine the gauge group

$$G = SU(k_1) \times \dots \times SU(k_s) \times U(1)^{s-1} \simeq \frac{U(k_1) \times \dots \times U(k_s)}{U(1)_{\text{Diag}}} , \quad (6.24)$$

where any $SU(1)$ factors must be dropped and $U(1)_{\text{Diag}}$ is the diagonal subgroup of $U(k_1) \times \dots \times U(k_s)$. Secondly, for any i with $k_i > 0$ there are $g(\tilde{\mathcal{C}}_i)$ four-dimensional $\mathcal{N} = 2$ hypermultiplets in the adjoint representation of $SU(k_i)$. Thirdly, we have $\tilde{\chi}_{ij}$ four-dimensional $\mathcal{N} = 2$ hypermultiplets in the bi-fundamental representation

$(\mathbf{k}_i, \mathbf{k}_j)_{(+1_i, +1_j)}$ of the gauge group factors $SU(k_1) \times SU(k_1)$, where the subscripts indicate the $U(1)$ -charges with respect to the diagonal $U(1)_i$ and $U(1)_j$ subgroups of the respective unitary groups $U(k_i)$ and $U(k_j)$ in the relation (6.24). The multiplicities $\tilde{\chi}_{ij}$ are again determined by the intersection numbers of the curves $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$ in the K3 fiber S . The resulting gauge theory spectrum is summarized in Table 6.2.

From the described spectrum and the results of ref. [41] we are now ready to analyze the branches of the $\mathcal{N} = 2$ gauge theory sectors. First, we determine the complex dimension h^b of the Higgs branch

$$h^b = \dim_{\mathbb{C}} H^b = 2 \left(\sum_{i=1}^s (g(\tilde{\mathcal{C}}_i) - 1)(k_i^2 - 1) \right) + 2 \left(\sum_{1 \leq i < j \leq s} \tilde{\chi}_{ij} k_i k_j \right) - 2(s-1). \quad (6.25)$$

Here, the first term captures the $2(k_i^2 - 1)$ complex degrees of freedom of the four-dimensional $\mathcal{N} = 2$ hypermultiplets in the corresponding adjoint representations of the $SU(k_i)$ gauge group factors — reduced by one adjoint $\mathcal{N} = 2$ Goldstone hypermultiplet rendering the four-dimensional $\mathcal{N} = 2$ $SU(k_i)$ vector multiplet massive. The second term realizes the complex degrees of freedom of the four-dimensional $\mathcal{N} = 2$ matter hypermultiplets in the bi-fundamental representations of the associated special unitary gauge groups and charged with respect to the appropriate $U(1)$ factors. The last term subtracts from the second term the $\mathcal{N} = 2$ Goldstone hypermultiplets for higgsing the $(s - 1)$ four-dimensional $\mathcal{N} = 2$ $U(1)$ vector multiplets.

Next, we derive the complex dimension of the Coulomb branch C^b , in which the maximal Abelian subgroup $U(1)^{n-1}$ remains unbroken. It is parameterized by the expectation value of all four-dimensional $\mathcal{N} = 2$ hypermultiplet components that are neutral with respect to this unbroken maximal Abelian subgroup. Therefore, the complex dimension c^\sharp of the Coulomb branch becomes

$$c^\sharp = \dim_{\mathbb{C}} C^\sharp = 2 \left(\sum_{i=1}^s g(\tilde{\mathcal{C}}_i)(k_i - 1) \right) + (n - 1). \quad (6.26)$$

The first term counts the traceless neutral diagonal degrees of freedom of the four-dimensional $\mathcal{N} = 2$ matter hypermultiplets in the adjoint representation, while the second term adds the contributions of the complex scalar fields in the four-dimensional unbroken Abelian $\mathcal{N} = 2$ vector multiplets.

The next task is to compute the Betti numbers of the twisted connected sum G_2 -manifolds Y^b and Y^\sharp , which geometrically realize the Higgs and Coulomb branch by orthogonal gluing of the building blocks (Z^b, S^b) and (Z^\sharp, S^\sharp) to a common right building block (Z_R, S_R) . We construct the building block (Z^b, S^b) by blowing-up the semi-Fano threefold P along the smooth irreducible curve \mathcal{C}^b , which — as in the Higgs branch of the Abelian gauge theories — arises from a generic deformation of the section s_0 of the anti-canonical line bundle $-K_P$. Then the relations (6.15) determine again the two-form and three-form Betti numbers of the G_2 -manifold Y^b . The smooth

Coulomb branch building block (Z^\sharp, S^\sharp) results from the sequence of $n = k_1 + \dots + k_s$ blow-ups

$$Z^\sharp = \text{Bl}_{\{\tilde{\mathcal{C}}_1^{k_1}, \dots, \tilde{\mathcal{C}}_s^{k_s}\}} P, \quad (6.27)$$

where each individual curve $\tilde{\mathcal{C}}_i$ is resolved k_i times such that $\dim k^\sharp = n - 1$. Therefore, using eqs. (5.11), (6.5) and (6.6), we arrive for the smooth G_2 -manifold Y^\sharp at the Betti numbers

$$b_2(Y^\sharp) = (n - 1) + \delta_R^{(2)}, \quad b_3(Y^\sharp) = b_3(P) + \left(\sum_{i=1}^s k_i \tilde{\mathcal{C}}_i \cdot \tilde{\mathcal{C}}_i \right) + 3n + 22 + \delta_R^{(3)}, \quad (6.28)$$

with the definitions (6.17). Using the equivalence relation $\mathcal{C}^\flat \sim k_1 \tilde{\mathcal{C}}_1 + \dots + k_s \tilde{\mathcal{C}}_s$, we calculate the change of Betti numbers

$$\begin{aligned} b_2(Y^\flat) &= b_2(Y^\sharp) - (n - 1), \\ b_3(Y^\flat) &= b_3(Y^\sharp) + \left(\sum_{i=1}^s \tilde{\chi}_{ii} k_i (k_i - 1) \right) + 2 \left(\sum_{1 \leq i < j \leq s} \tilde{\chi}_{ij} k_i k_j \right) - 3(n - 1). \end{aligned} \quad (6.29)$$

in terms of the intersection numbers $\tilde{\chi}_{ij} = \tilde{\mathcal{C}}_i \cdot \tilde{\mathcal{C}}_j$ on the K3 surface S .

As for the Abelian gauge theory sectors, the computed change of Betti numbers is also in accord with the phase structure of the proposed non-Abelian gauge theory description. Namely, the change of the two-form Betti number conforms with the difference of the four-dimensional $\mathcal{N} = 1$ vector multiplets in the Higgs and Coulomb branches, given by the rank of the non-Abelian gauge group (6.24). The difference of four-dimensional $\mathcal{N} = 1$ chiral multiplets is accurately predicted by the complex dimensions of the Higgs and Coulomb branches. That is to say with eqs. (6.6), (6.25) and (6.26) we find for the discussed non-Abelian gauge theories

$$b_3(Y^\flat) - b_3(Y^\sharp) = b_3^\flat - b_3^\sharp = \dim_{\mathbb{C}} H^\flat - \dim_{\mathbb{C}} C^\sharp. \quad (6.30)$$

As for the previously discussed Abelian gauge theories, the established correspondence between G_2 -manifolds and non-Abelian Higgs and Coulomb branches carries over to mixed Higgs–Coulomb branches as well, which we illustrate with an explicit example in Section 6.3. The fact that the performed analysis of the non-Abelian gauge theory sectors closely parallels the study of the Abelian gauge theories does not come as a surprise, because the Abelian gauge group (6.10) arises from partially higgsing the non-Abelian gauge group (6.24) to its maximal Abelian subgroup. As a result, the topological data of the G_2 -manifolds for the Higgs, Coulomb and mixed Higgs–Coulomb phases resulting from a given semi-Fano threefold P are the same for both the discussed Abelian and non-Abelian gauge theory sectors.

6.3 Examples of G_2 -manifolds with $\mathcal{N} = 2$ gauge theories

Following the general discussion of $\mathcal{N} = 2$ gauge theory sectors in Section 6.1 and Section 6.2, we now illustrate the emergence of $\mathcal{N} = 2$ gauge theory sectors in twisted connected sum G_2 -manifolds with a few explicit examples:

$SU(4)$ gauge theory with adjoint matter from the Fano threefold \mathbb{P}^3 : Consider the Fano threefold \mathbb{P}^3 with the anti-canonical divisor $-K_{\mathbb{P}^3} = 4H$ in terms of the hyperplane class H . Let $\tilde{s}_{0,1}$ and s_1 be a (generic) global section of H and $-K_{\mathbb{P}^3}$, respectively. Then we obtain with eq. (6.22) the resolved building block $Z_{\text{sing}} \subset \mathbb{P}^3 \times \mathbb{P}^1$ as the hypersurface equation

$$\tilde{s}_{0,1}^4 + ts_1 = 0, \quad (6.31)$$

with the affine coordinate t of the factor \mathbb{P}^1 . This equation exhibits an A_3 singularity along the curve $\tilde{\mathcal{C}}_1 = \{\tilde{s}_{0,1} = 0\} \cap \{s_1 = 0\} \cap \{t = 0\}$, which yields a $\mathcal{N} = 2$ gauge theory sector with gauge group $SU(4)$. Note that for this particular example the deformed phases of the non-enhanced $\mathcal{N} = 2$ Abelian gauge theory sector are discussed as well in ref. [25].

We first note that the curves $\mathcal{C}^{(k)} = (-K_{\mathbb{P}^3}) \cap (kH)$ have the following intersection numbers on the K3 surface S and — according to eq. (6.6) — genera

$$\mathcal{C}^{(k)} \cdot \mathcal{C}^{(l)} = 4kl, \quad g(\mathcal{C}^{(k)}) = \frac{1}{2} \mathcal{C}^{(k)} \cdot \mathcal{C}^{(k)} + 1 = 2k^2 + 1. \quad (6.32)$$

Due to the equivalence $\tilde{\mathcal{C}}_1 \sim \mathcal{C}^{(k)}$ we arrive at $g(\tilde{\mathcal{C}}_1) = 3$ four-dimensional $\mathcal{N} = 2$ hypermultiplets in the adjoint representation of $SU(4)$. This spectrum predicts with eqs. (6.25) and (6.24) the dimensions of the Higgs and Coulomb branches

$$\dim_{\mathbb{C}} H^{\flat} = 60, \quad \dim_{\mathbb{C}} C^{\sharp} = 21, \quad \dim_{\mathbb{C}} H^{\flat} - \dim_{\mathbb{C}} C^{\sharp} = 39. \quad (6.33)$$

As proposed in Section 6.2 by sequentially blowing-up \mathbb{P}^3 four times along the curve $\tilde{\mathcal{C}}_1$, we arrive at the building block (Z^{\sharp}, S^{\sharp}) with

$$\dim k^{\sharp} = 3, \quad b_3(Z^{\sharp}) = 4 \cdot 2g(\tilde{\mathcal{C}}_1) = 24. \quad (6.34)$$

Deforming the hypersurface equation (6.31) to $s_0 + ts_1 = 0$ with a generic section of $-K_{\mathbb{P}^3}$, we resolve along the reduced smooth curve $\mathcal{C}^{\flat} \subset \mathbb{P}^3$ with $\mathcal{C}^{\flat} \sim \mathcal{C}^{(4)}$ in order to determine the building block (Z^{\flat}, S^{\flat}) of the Higgs branch H^{\flat} with

$$\dim k^{\flat} = 0, \quad b_3(Z^{\flat}) = 2g(\mathcal{C}^{\flat}) = 66. \quad (6.35)$$

Finally, orthogonally gluing the building blocks (Z^{\flat}, S^{\flat}) and (Z^{\sharp}, S^{\sharp}) to a suitable right building block (Z_R, S_R) , we obtain with eq. (5.11) the twisted connected G_2 -manifolds Y^{\flat} and Y^{\sharp} with the reduced Betti numbers

$$\begin{aligned} b_2^{\flat} &= 0, & b_3^{\flat} &= 89, \\ b_2^{\sharp} &= 3, & b_3^{\sharp} &= 50, \end{aligned} \quad (6.36)$$

s_0 factors	Gauge Group	$\mathcal{N} = 2$ Hypermultiplet spectrum	h^\flat	c^\sharp	b_3^\flat	b_3^\sharp	k^\sharp
1^4	$SU(4)$	$3 \times \mathbf{adj}$	60	21	89	50	3
$1^3 \cdot 1$	$SU(3) \times U(1)$	$3 \times \mathbf{adj}; 4 \times \mathbf{3}_{+1}$	54	15	89	50	3
$1^2 \cdot 1^2$	$SU(2)^2 \times U(1)$	$3 \times (\mathbf{adj}, 1); 3 \times (1, \mathbf{adj}); 4 \times (\mathbf{2}, \mathbf{2})_{+1}$	54	15	89	50	3
$1^2 \cdot 1 \cdot 1$	$SU(2) \times U(1)^2$	$3 \times \mathbf{adj}; 4 \times \mathbf{2}_{(+1,+1)}; 4 \times \mathbf{2}_{(+1,0)}; 4 \times \mathbf{2}_{(0,+1)}$	48	9	89	50	3
$1 \cdot 1 \cdot 1 \cdot 1$	$U(1)^3$	$4 \times (+1, +1, 0); 4 \times (+1, 0, +1); 4 \times (0, +1, +1);$ $4 \times (+1, 0, 0); 4 \times (0, +1, 0); 4 \times (0, 0, +1)$	42	3	89	50	3
$2 \cdot 1^2$	$SU(2) \times U(1)$	$3 \times \mathbf{adj}; 8 \times \mathbf{2}_{+1}$	42	8	89	55	2
$2 \cdot 1 \cdot 1$	$U(1)^2$	$4 \times (+1, +1); 8 \times (+1, 0); 8 \times (0, +1)$	36	2	89	55	2
2^2	$SU(2)$	$9 \times \mathbf{adj}$	48	19	89	60	1
$2 \cdot 2$	$U(1)$	$16 \times (+1)$	30	1	89	60	1
$3 \cdot 1$	$U(1)$	$12 \times (+1)$	22	1	89	68	1

Table 6.3: Depicted are the gauge theory branches of the $SU(4)$ gauge theory of the building blocks associated to the rank one Fano threefold \mathbb{P}^3 . The columns list the factorization of the anti-canonical section s_0 with degrees and multiplicities, the gauge group of the gauge theory branch, the matter spectrum of $\mathcal{N} = 2$ hypermultiplets with their non-Abelian representations together with the Abelian $U(1)$ charges, the complex dimensions h^\flat and c^\sharp of the Higgs and Coulomb branches, the reduced three-form Betti numbers b_3^\flat and b_3^\sharp of the twisted connected G_2 -manifolds Y^\flat and Y^\sharp , and the kernel k^\sharp of the Coulomb phase building block (Z^\sharp, S^\sharp) .

which we defined in eq. (6.18). We observe that the differences $b_2^\sharp - b_2^\flat = 3$ and $b_3^\flat - b_3^\sharp = 39$ agree with the rank of the gauge group and the change in dimensionality of the Higgs and Coulomb branches, respectively, which is in accord with the anticipated gauge theory description established in Section 6.2.

By partially deforming the first term $\tilde{s}_{0,1}^4$ in the hypersurface equation (6.31), we can realize hypersurface singularities describing various Abelian and non-Abelian subgroups of $SU(4)$. Such partial deformations geometrically realize mixed Higgs–Coulomb branches of the $SU(4)$ gauge theory. We collect the geometry and phase structure of these mixed Higgs–Coulomb branches in Table 6.3, where the entries of this table are determined with eqs. (6.4), (6.5), (6.25), (6.26), and (6.32). Note that — depending on the breaking pattern of $SU(4)$ arising from partially higgsing — the dimensions of Higgs and Coulomb branches vary because only the charged matter spectrum of the unbroken gauge group plays a role for the Higgs and Coulomb branches in this gauge theory sector. For all entries in Table 6.3 we find that

$$b_3^\flat - b_3^\sharp = h^\flat - c^\sharp, \quad \dim k^\sharp = \text{rk } G. \quad (6.37)$$

This agreement confirms nicely the correspondence between gauge theory branches

s_0 factors	Gauge Group	$\mathcal{N} = 2$ Hypermultiplet spectrum	h^b	c^\sharp	b_3^b	b_3^\sharp	k^\sharp
$(1, 0)^2(0, 1)^2$	$SU(2) \times SU(2) \times U(1)$	$2 \times (\mathbf{adj}, \mathbf{1}); 2 \times (\mathbf{1}, \mathbf{adj}); 4 \times (\mathbf{2}, \mathbf{2})_{+1}$	42	11	50	19	3
$(1, 0)^2(0, 1)(0, 1)$	$SU(2) \times U(1)^2$	$2 \times \mathbf{adj}; 4 \times \mathbf{2}_{(1,0)}; 4 \times \mathbf{2}_{(0,1)}; 2 \times \mathbf{1}_{(1,1)}$	38	7	50	19	3
$(1, 0)(1, 0)$ $\cdot (0, 1)(0, 1)$	$U(1)^3$	$2 \times (1, 1, 0); 4 \times (1, 0, 1); 4 \times (0, 1, 1);$ $4 \times (1, 0, 0); 4 \times (0, 1, 0); 2 \times (0, 0, 1)$	34	3	50	19	3
$(2, 0)(0, 1)^2$	$SU(2) \times U(1)$	$2 \times \mathbf{adj}; 8 \times \mathbf{2}_{+1}$	36	6	50	20	2
$(2, 0)(0, 1)(0, 1)$	$U(1)^2$	$8 \times (1, 0); 8 \times (0, 1); 2 \times (1, 1)$	32	2	50	20	2
$(1, 1)^2$	$SU(2)$	$7 \times \mathbf{adj}$	36	15	50	29	1
$(1, 1)(1, 1)$	$U(1)$	$12 \times (+1)$	22	1	50	29	1
$(2, 0)(0, 2)$	$U(1)$	$16 \times (+1)$	30	1	50	21	1
$(2, 1)(0, 1)$	$U(1)$	$10 \times (+1)$	18	1	50	33	1

Table 6.4: The table shows the branches of the $SU(2) \times SU(2) \times U(1)$ gauge theory associated to the Fano threefold W_6 with Mori–Mukai label MM48₂ [62]. Listed are the factors of the anti-canonical section s_0 with bi-degrees and multiplicities, the unbroken gauge subgroup, the $\mathcal{N} = 2$ matter hypermultiplets, the complex dimensions h^b and c^\sharp of the Higgs and Coulomb branches, the reduced three-form Betti numbers b_3^b and b_3^\sharp of the twisted connected G_2 -manifolds Y^b and Y^\sharp , and the kernel k^\sharp of the Coulomb phase.

and phases of twisted connected G_2 -manifolds.

$SU(2) \times SU(2) \times U(1)$ gauge theory from the Fano threefold MM48₂: The rank two Fano threefold W_6 with reference number MM48₂ is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with $b_3(W_6) = 0$ [62]. Let H_1 and H_2 be the hyperplane classes of $\mathbb{P}^2 \times \mathbb{P}^2$. Then by adjunction the anti-canonical divisor of W_6 reads $-K_{W_6} = 2H_1 + 2H_2$. Furthermore, the self-intersection numbers of the curves $\mathcal{C}^{(k,l)} = (-K_{W_6}) \cap (kH_1 + lH_2)$ in the anti-canonical divisor $-K_{W_6}$ and hence their genera are

$$\mathcal{C}^{(k_1, l_1)} \cdot \mathcal{C}^{(k_2, l_2)} = 2(k_1 k_2 + l_1 l_2 + 2k_1 l_2 + 2l_1 k_2), \quad g(\mathcal{C}^{(k, l)}) = k^2 + l^2 + 4kl + 1. \quad (6.38)$$

With generic global sections $\tilde{s}_{0,1}$, $\tilde{s}_{0,2}$ and s_1 of H_1 , H_2 and $-K_{W_6}$ the equation for the singular building block $Z_{\text{sing}} \subset W_6 \times \mathbb{P}^1$ becomes with the affine coordinate t of \mathbb{P}^1

$$\tilde{s}_{0,1}^2 \tilde{s}_{0,2}^2 + t s_1 = 0. \quad (6.39)$$

Thus, we find A_1 singularities along the two curves $\tilde{\mathcal{C}}_i = \{\tilde{s}_{0,i} = 0\} \cap \{s_1 = 0\} \cap \{t = 0\}$ with $i = 1, 2$. Thus — following the general discussion in Section 6.2 — we find a $SU(2) \times SU(2) \times U(1)$ gauge theory both with adjoint matter and with bi-fundamental matter from the intersection points $\tilde{\mathcal{C}}_1 \cap \tilde{\mathcal{C}}_2$. Due to the equivalences $\tilde{\mathcal{C}}_1 \sim \mathcal{C}^{(1,0)}$

and $\tilde{\mathcal{C}}_1 \sim \mathcal{C}^{(0,1)}$ and the relations (6.38), we arrive at the four-dimensional $\mathcal{N} = 2$ hypermultiplet matter spectrum

$$2 \times (\mathbf{adj}, \mathbf{1}) ; \quad 2 \times (\mathbf{1}, \mathbf{adj}) ; \quad 4 \times (\mathbf{2}, \mathbf{2})_{+1} . \quad (6.40)$$

The resulting correspondence between the gauge theory branches and the phase structure of the twisted connected G_2 -manifold is summarized in Table 6.4, where the entries are computed with the formulas (6.4), (6.5), (6.25), (6.26), and (6.38).

Further examples from toric semi-Fano threefolds: Our last class of examples concerns $\mathcal{N} = 2$ gauge theory sectors from toric semi-Fano threefolds P_Σ , where the fan Σ is obtained from a triangulation of a three-dimensional reflexive lattice polytope Δ . In this toric setup the anti-canonical divisor reads

$$-K_{P_\Sigma} = D_1 + \dots + D_n , \quad (6.41)$$

where the toric divisors D_i correspond to the one-dimensional cones of Σ , that is to say to the rays of the lattice polytope Δ . For smooth toric varieties P_Σ the toric divisors D_i are smooth and intersect transversely [59]. As the anti-canonical divisor $-K_P$ is base point free we can apply Bertini's theorem — see, e.g., ref. [77] — to argue that we can find a smooth global section s_1 of the anti-canonical divisor $-K_{P_\Sigma}$ and further generic global sections $s_{0,i}$ of D_i such that the curves $\mathcal{C}_i = \{s_{0,i} = 0\} \cap \{s_1 = 0\}$ are smooth and mutually intersect transversely. Hence, the toric semi-Fano threefold P_Σ realizes indeed a $U(1)^{n-1}$ gauge theory sector. The four-dimensional matter spectrum is then given by Table 6.1, where the multiplicities χ_{ij} are the toric triple intersection numbers

$$\chi_{ij} = -K_{P_\Sigma} \cdot D_i \cdot D_j . \quad (6.42)$$

As proposed in Section 6.1 we construct the building blocks (Z^\sharp, S^\sharp) of the Coulomb branch C^\sharp by the sequential blow-ups (6.3) along the curves \mathcal{C}_i , while we determine the building block (Z^b, S^b) of the Higgs branch H^b by blowing a smooth curve $\mathcal{C}^b = \{s_0 = 0\} \cap \{s_1 = 0\}$ obtained by deforming the singular section $s_{0,1} \cdots s_{0,n}$ to generic anti-canonical section s_0 . Then we arrive at the twisted connected G_2 -manifolds Y^\sharp and Y^b by orthogonally gluing these gauge theory building blocks with a right building block (Z_R, S_R) in the usual way.³³

Note that due to linear equivalences among the toric divisors D_i the Abelian gauge theory can enhance to non-Abelian gauge groups as well. Namely, assume that the anti-canonical bundle $-K_{P_\Sigma}$ is linearly equivalent to

$$-K_{P_\Sigma} \sim k_1 \tilde{D}_1 + \dots + k_s \tilde{D}_s , \quad (6.43)$$

³³For toric semi-Fano threefolds P_Σ some of the performed blow-ups discussed here and in the following can also be described with toric geometry techniques [26]. However, such a toric description is not advantageous to extract the relevant geometric data for us.

where for some divisors $\tilde{D}_\alpha \sim \sum_i a_{\alpha i} D_i$ with global sections $\tilde{s}_{0,\alpha}$. Furthermore, we require that the curves \tilde{C}_α are smooth and mutually intersect transversely. Then following Section 6.2 we arrive at the $\mathcal{N} = 2$ gauge theory sector with gauge group

$$G = SU(k_1) \times \dots \times SU(k_s) \times U(1)^{s-1} . \quad (6.44)$$

Note that rank of the gauge group $\tilde{n} = k_1 + \dots + k_s - 1$ is a priori not correlated with the number n of toric divisors. Instead, it depends on the precise nature of the linear equivalences among the toric divisors D_i , $i = 1, \dots, n$, and the divisors \tilde{D}_α , $i = 1, \dots, s$.

Let us exemplify the study of four-dimensional $\mathcal{N} = 2$ gauge theory sectors with the rank two toric semi-Fano threefold P_Σ with the reference number K32 described in some detail in Section 5.3. Using for this example the linear equivalences among the toric divisors D_1, \dots, D_5 , c.f., below eq. (5.14), we find for the anti-canonical divisor

$$-K_{P_\Sigma} = D_1 + \dots + D_5 \sim 3D_1 \sim 3D_2 + 3D_4 . \quad (6.45)$$

With these linearly equivalent representations for $-K_{P_\Sigma}$, we for instance arrive at the gauge groups $U(1)^4$ of rank four, $SU(3)$ of rank three, or $SU(3) \times SU(3) \times U(1)$ of rank five. Note that the phases of the lower rank gauge groups $U(1)^4$ and $SU(3)$ enjoy again the interpretation as mixed Higgs–Coulomb branches of the $SU(3) \times SU(3) \times U(1)$ gauge theory of rank five, which by applying eq. (6.42) and eq. (6.6) yields the spectrum

$$1 \times (\mathbf{adj}, \mathbf{1}) ; \quad 1 \times (\mathbf{1}, \mathbf{adj}) ; \quad 3 \times (\mathbf{3}, \mathbf{3})_{+1} . \quad (6.46)$$

In Table 6.5 we list the gauge theory sectors of a few toric semi-Fano threefolds P_Σ . This table does not display all mixed Higgs–Coulomb branches. Here, we focus on the resulting twisted G_2 -manifolds Y^\flat and Y^\sharp associated to the Higgs H^\flat and Coulomb branches C^\sharp of the maximally enhanced gauge group of maximal rank, as obtained by the factorization of the anti-canonical bundle $-K_{P_\Sigma}$.

No.	ρ	Gauge Group	$\mathcal{N} = 2$ Hypermultiplet spectrum	h^\flat	c^\sharp	b_3^\flat	b_3^\sharp	k^\sharp
K24, MM34 ₂	2	$SU(3) \times SU(2)$ $\times U(1)$	$2 \times (\mathbf{adj}, 1); (1, \mathbf{adj}); 3 \times (\mathbf{3}, \mathbf{2})_{+1}$	50	14	79	43	4
K32	2	$SU(3)^2 \times U(1)$	$(\mathbf{adj}, 1); (1, \mathbf{adj}); 3 \times (\mathbf{3}, \mathbf{3})_{+1}$	52	13	79	40	5
K35, MM36 ₂	2	$SU(5) \times SU(2)$ $\times U(1)$	$2 \times (\mathbf{adj}, 1); (\mathbf{5}, \mathbf{2})_{+1}$	60	22	87	49	6
K36, MM35 ₂	2	$SU(4) \times SU(2)$ $\times U(1)$	$2 \times (\mathbf{adj}, 1); 2 \times (\mathbf{4}, \mathbf{2})_{+1}$	54	17	81	44	5
K37, MM33 ₂	2	$SU(4) \times SU(3)$ $\times U(1)$	$(\mathbf{adj}, 1); 3 \times (\mathbf{4}, \mathbf{3})_{+1}$	54	12	79	37	6
K62, MM27 ₃	3	$SU(2)^3 \times U(1)^2$	$(\mathbf{adj}, 1^2); (1, \mathbf{adj}, 1); (1^2, \mathbf{adj}); 2 \times (\mathbf{2}^2, 1)_{(1,1)}$ $2 \times (\mathbf{2}, 1, \mathbf{2})_{(1,0)}; 2 \times (\mathbf{1}, \mathbf{2}^2)_{(0,1)}$	44	11	73	40	5
K68, MM25 ₃	3	$SU(3) \times SU(2)$ $\times U(1)^2$	$(\mathbf{adj}, 1); 3 \times (\mathbf{3}, \mathbf{2})_{(1,1)}; 2 \times (\mathbf{3}, \mathbf{1})_{(1,0)}; (\mathbf{1}, \mathbf{2})_{(0,1)}$	42	9	69	36	6
K105, MM31 ₃	3	$SU(3)^2 \times SU(2)$ $\times U(1)^2$	$(\mathbf{adj}, 1^2); (1, \mathbf{adj}, 1); 2 \times (\mathbf{3}^2, 1)_{(1,1)}; (\mathbf{3}, 1, \mathbf{2})_{(1,0)};$ $(\mathbf{1}, \mathbf{3}, \mathbf{2})_{(0,1)}$	50	15	77	42	7
K124	3	$SU(4) \times SU(2)^2$ $\times U(1)^2$	$(\mathbf{adj}, 1, 1); 2 \times (\mathbf{4}, \mathbf{2}, 1)_{(1,1)}; 2 \times (\mathbf{4}, 1, \mathbf{2})_{(1,0)}$	48	13	73	38	7
K218, MM12 ₄	4	$SU(4) \times SU(3)$ $\times SU(2)^2 \times U(1)^3$	$(\mathbf{adj}, 1^3); (\mathbf{4}, \mathbf{3}, 1^2)_{(1,1,0)}; (\mathbf{4}, 1, \mathbf{2}, 1)_{(1,0,1)};$ $(\mathbf{4}, 1^2, \mathbf{2})_{(1,0,0)}; (\mathbf{1}, \mathbf{3}, \mathbf{2}, 1)_{(0,1,1)}; (\mathbf{1}, \mathbf{3}, 1, \mathbf{2})_{(0,1,0)}$	46	16	71	41	10
K266, MM10 ₄	4	$SU(3) \times SU(2)^3$ $\times U(1)^3$	$(1, \mathbf{adj}, 1^2); (\mathbf{3}, \mathbf{2}, 1^2)_{(1,1,0)}; 2 \times (\mathbf{3}, 1, \mathbf{2}, 1)_{(1,0,1)};$ $2 \times (\mathbf{3}, 1^2, \mathbf{2})_{(1,0,0)}; (\mathbf{1}, \mathbf{2}^2, 1)_{(0,1,1)}; (\mathbf{1}, \mathbf{2}, 1, \mathbf{2})_{(0,1,0)}$	42	10	67	35	8
K221	4	$SU(3) \times SU(2)^2$ $\times U(1)^3$	$2 \times (\mathbf{3}, \mathbf{2}, 1)_{(1,1,0)}; 3 \times (\mathbf{3}, 1, \mathbf{2})_{(1,0,1)}; (\mathbf{3}, 1^2)_{(1,0,0)}$ $2 \times (\mathbf{1}, \mathbf{2}, 1)_{(0,1,0)}$	40	7	63	30	7
K232	4	$SU(4) \times SU(2)^3$ $\times U(1)^3$	$2 \times (\mathbf{4}, \mathbf{2}, 1^2)_{(1,1,0)}; 2 \times (\mathbf{4}, 1, \mathbf{2}, 1)_{(1,0,1)};$ $2 \times (\mathbf{4}, 1^2, \mathbf{2})_{(1,0,0)}$	42	9	65	32	9
K233	4	$SU(3) \times SU(2)^2$ $\times U(1)^2$	$3 \times (\mathbf{3}, \mathbf{2}, 1)_{(1,1)}; 3 \times (\mathbf{3}, 1, \mathbf{2})_{(1,0)}$	40	6	63	29	6
K247	4	$SU(4) \times SU(3)^2$ $\times SU(2) \times U(1)^3$	$2 \times (\mathbf{4}, \mathbf{3}, 1^2)_{(1,1,0)}; 2 \times (\mathbf{4}, 1, \mathbf{3}, 1)_{(1,0,1)};$ $(\mathbf{1}, \mathbf{3}, 1, \mathbf{2})_{(0,1,0)}; (\mathbf{1}^2, \mathbf{3}, \mathbf{4})_{(0,0,1)}$	46	11	69	34	11
K257	4	$SU(5) \times SU(3)^2$ $\times SU(2) \times U(1)^3$	$2 \times (\mathbf{5}, \mathbf{3}, 1^2)_{(1,1,0)}; 2 \times (\mathbf{5}, 1, \mathbf{3}, 1)_{(1,0,1)};$ $(\mathbf{5}, 1^2, \mathbf{2})_{(1,0,0)}$	48	12	71	35	12

Table 6.5: The table exhibits the $\mathcal{N} = 2$ gauge theory sectors for some smooth toric semi-Fano threefolds P_Σ of Picard rank two and higher. The columns display the number of the threefold P_Σ in the Mori–Mukai [62] and/or Kasprzyk [64] classification, its Picard rank ρ , the maximally enhanced gauge group of maximal rank by factorizing the anti-canonical bundle, the $\mathcal{N} = 2$ matter hypermultiplets, the complex dimensions h^\flat and c^\sharp of the Higgs and Coulomb branches, the reduced three-form Betti numbers b_3^\flat and b_3^\sharp , and the kernel k^\sharp of the Coulomb branch.

6.4 Transitions among twisted connected sum G_2 -manifolds

The proposed correspondence between phases of twisted connected G_2 -manifolds and gauge theory branches of the described $\mathcal{N} = 2$ gauge theory sectors is essentially based upon the correspondence between extremal transitions in the asymptotically cylindrically Calabi–Yau threefolds $X_{L/R}$ and the Higgs–Coulomb phase structure of the associated $\mathcal{N} = 2$ gauge theories. In the original type IIA string theory setting the $\mathcal{N} = 2$ matter spectrum arises from solitons of massless D2-branes wrapping the vanishing cycles of the singular Calabi–Yau threefolds at the transition point [39, 38], which become membranes in the discussed context of M-theory. However, while in the type IIA setting these D2-branes furnish BPS states of the $\mathcal{N} = 2$ algebra, the corresponding interpretation of membrane states becomes more subtle in the context of M-theory on twisted connected sum G_2 -manifolds because the corresponding membrane states do not admit a BPS interpretation due to minimal four-dimensional $\mathcal{N} = 1$ supersymmetry. Therefore, a natural question now is if the described M-theory transitions are actually dynamically realized.

As discussed in Section 2 the semi-classical moduli space $\mathcal{M}_{\mathbb{C}}$ of M-theory on G_2 -manifolds has the geometric moduli space \mathcal{M} of Ricci-flat G_2 -manifolds as a real subspace. From the low-energy effective $\mathcal{N} = 1$ supergravity point of view this is a consequence of the semi-classical shift symmetries with respect to the real parts of the chiral fields (2.31). However, due to arguments about the absence of global continuous symmetries in consistent theories of gravity, see e.g., ref. [78], these shift symmetries should be broken non-perturbatively such that the flat directions of the chiral moduli fields are lifted. In the context of M-theory on G_2 -manifolds membrane instantons on suitable three-cycles generate non-perturbative superpotential terms that break these continuous shift symmetries [22]. As these non-perturbative corrections are exponentially suppressed in the volume of the wrapped three-cycles, the flat directions — as described by the semi-classical moduli space $\mathcal{M}_{\mathbb{C}}$ — are expected to be only realized in the large volume limit of the G_2 -compactification. Hence, M-theory transitions among G_2 -manifolds should only occur in the absence of such non-perturbative effects, as is for instance the case in the large volume limit.³⁴

If we now take both the large volume limit and the Kovalev limit simultaneously, gravity decouples, and we arrive in this limit at a genuine four-dimensional $\mathcal{N} = 2$ gauge theory sector with eight unbroken supercharges. Then the lower energy dynamics is indeed described as in refs. [39, 38, 40, 41], and the gauge theory phases connect asymptotically cylindrical Calabi–Yau threefolds via extremal transitions. Thus, we claim that in the large volume and in the large Kovalev limit the transitions among the $\mathcal{N} = 2$ gauge theory sectors geometrically realize the anticipated transitions among twisted connected sum G_2 -manifolds.

³⁴In the presence of small non-perturbative obstructions we can still have quantum-mechanical transitions among four-dimensional vacua. Then the transition probability is governed by the tunneling rate through the barrier of the non-perturbative scalar potential.

If we maintain the large volume limit but allow for finite Kovalevton the situation becomes more subtle. While the massless spectrum is still $\mathcal{N} = 2$, we expect that the appearance of further interaction terms break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. Then the $\mathcal{N} = 2$ gauge theory sector is partially broken to a $\mathcal{N} = 1$ gauge theory, whose supersymmetry breaking couplings are governed by the scale of the Kovalevton. In this $\mathcal{N} = 1$ language the transition between non-Abelian $\mathcal{N} = 2$ Higgs and Coulomb branches essentially describes an enhancement to an Abelian gauge symmetry within the $\mathcal{N} = 1$ Higgs branches. Namely, in the $\mathcal{N} = 1$ language the $\mathcal{N} = 2$ Coulomb phase corresponds to the partially higgsing of the non-Abelian group to its maximal Abelian subgroup. Thus, at low energies the proposed (non-Abelian) $\mathcal{N} = 2$ Higgs–Coulomb phase transition describes the Higgs mechanism of a weakly-coupled Abelian $\mathcal{N} = 1$ gauge theory. These observations provide for some evidence that in the large volume limit the anticipated phase structure among the described twisted connected sum G_2 -manifolds is still realized — even for finite Kovalevton.

Geometrically, we therefore propose that in the M-theory moduli space $\mathcal{M}_{\mathbb{C}}$ the presented transitions among twisted connected sum G_2 -manifolds are indeed unobstructed. That is to say, we conjecture that the construction of orthogonally gluing commutes with extremal transitions in the asymptotically cylindrical Calabi–Yau threefolds $X_{L/R}$. Furthermore, our proposal implies that the moduli space \mathcal{M} of Ricci-flat G_2 -metrics of the twisted connected sum type should exhibit a stratification structure as predicted by the phase structure of the analyzed $\mathcal{N} = 2$ gauge theories sectors. In the context of Abelian gauge theory sectors our proposal conforms with a similar conjecture put forward in ref. [25].

7 Conclusions

In this work we have studied the four-dimensional low-energy effective $\mathcal{N} = 1$ supergravity action arising from M-theory compactified on G_2 -manifolds of the Kovalev’s twisted connected sum type. By suitably gluing a pair of non-compact asymptotically cylindrical Calabi–Yau threefolds times a circle $Y_{L/R} = X_{L/R} \times S^1$ in their asymptotic regions [23], this construction realizes a large class of examples for compact G_2 -manifolds [24–26], which are yet of a specific type realizing only a particular (non-trivial) homotopy invariant of G_2 -manifolds [29].

From the cohomology of such G_2 -manifolds, we established that this class of M-theory compactifications yields two neutral universal $\mathcal{N} = 1$ chiral moduli fields associated to the complexified overall volume modulus ν and the gluing modulus — called the Kovalevton \varkappa — respectively. The latter parametrizes the Kovalev limit taken by $\text{Re}(\varkappa) \rightarrow \infty$.

The proper interpretation of the different contributions (4.4) to the cohomology of the G_2 -manifold Y implies that there is a decomposition of the fields of the $\mathcal{N} = 1$ effective supergravity theory on Y into $\mathcal{N} = 1$ neutral chiral moduli multiplets, into

two $\mathcal{N} = 2$ gauge theory sectors coming from the two asymptotic regions $Y_{L/R}$, and into one $\mathcal{N} = 4$ gauge sector that comes from the trivial $K3$ fibration with fibre S in the gluing region $T^2 \times S \times (0, 1)$, c.f., Table 4.1. This decomposition into these gauge theory sectors becomes exact in a controllable way in the Kovalev limit and yields a scheme, in which the four-dimensional low energy effective theory can be approximated in terms of these sectors, with small corrections. In particular, we worked out the dependence on these two universal chiral moduli fields in the four-dimensional low-energy effective $\mathcal{N} = 1$ supergravity action resulting from a compactification on a smooth twisted connected sum G_2 -manifold Y . Moreover, the obtained two scales do also control behavior of M -theory corrections. Let us now list mathematical, physical and eventually even phenomenological prospects of these decomposition specific to M theory compactifications on twisted connected sum G_2 -manifolds.

A first consequence is that we can identify Abelian and non-Abelian gauge theory enhancements with various matter content from singularities in the asymptotic cylindrical Calabi–Yau threefolds $X_{L/R}$ in codimension four and six that occur in the twisted connected sum Y away from the gluing region. These lead to transitions in the threefolds $X_{L/R}$, whose deformations and resolutions can be described by methods of algebraic geometry familiar in the context of $\mathcal{N} = 2$ theories. The significant point established in Section 6 is that these transitions commute with the Kovalev limit and the gluing construction. Namely, they connect G_2 -manifolds whose change in the cohomology groups corresponds exactly to the change in the spectrum of $\mathcal{N} = 1$ vector and chiral superfields as predicted by the transitions. Concretely, starting with the equations that describe the blow-up of the anti-canonical divisor in semi-Fano threefolds and analyzing all their possible degenerations leads to a great variety of gauge groups and matter spectra as well as to many novel examples of twisted connected sum G_2 -manifolds corresponding to the different branches of these gauge theories.

This suggest that in a suitably compactified moduli space of the Ricci-flat G_2 metrics there are many new types of singular loci through which it is possible to reach topological inequivalent G_2 -manifolds. This question is a priori independent of the possible $\mathcal{N} = 1$ non-perturbative membrane instanton corrections that could lift the flat directions in the $\mathcal{N} = 1$ scalar potential (which are protected in the pure $\mathcal{N} = 2$ limit). Therefore, by taking the large volume and the Kovalev limit these directions certainly remain flat. However, we argued in Section 6.4 that even for finite expectation values of the Kovalevton $\not\propto$ the transitions should remain physical in the effective $\mathcal{N} = 1$ theory, as long as non-perturbative membrane instantons remain suppressed.

Another interesting physical consequence of the decomposition and the Kovalev limit is that the more advanced $\mathcal{N} = 2$ techniques — like calculating the exact gauge coupling and the exact BPS masses from the periods of the holomorphic three-form — are approached in this limit and serve as a zeroth order approximation with inverse power laws or exponential corrections in the Kovalevton $\not\propto$ and the volume modu-

lus ν , similarly as the calculations carried out in refs. [36, 69] in the context of local G_2 -manifolds. Those corrections leading to holomorphic terms in the four-dimensional $\mathcal{N} = 1$ effective theory are expected to be accessible by techniques similar to the ones used to calculate four-dimensional $\mathcal{N} = 1$ F-terms in flux and/or brane compactifications of type II theories. Note, however, that these computations would require a detailed study of the relative Calabi–Yau three-form periods on the two non-compact Calabi–Yau threefolds — for instance by using variation of mixed Hodge structure techniques along the lines of refs. [79–85] — and a moduli-dependent analysis of the matching conditions (3.9).³⁵

An attractive feature of the twisted connected sum compactification is that in the two individual $\mathcal{N} = 2$ gauge theory sectors from $X_{L/R}$ we have algebraic methods to geometrically engineer gauge groups, spectra and interactions. Already the few examples presented in Table 6.5 yield small rank gauge groups such as the standard model group and possible grand unification scenarios. The matter contents could in principle be broken into phenomenologically more suitable massless $\mathcal{N} = 1$ chiral matter multiplets. In fact the computed $\mathcal{N} = 2$ spectra can be broken to $\mathcal{N} = 1$ multiplets by various non-local effects. As discussed for finite Kovalevton \asymp the twisted gluing itself and non-perturbative effects — such as membrane instantons — introduce genuine $\mathcal{N} = 1$ interaction terms. Adding a flux-induced superpotential (2.36) offers yet another attractive mechanism to break the $\mathcal{N} = 2$ spectra into $\mathcal{N} = 1$ multiplets [8], potentially introducing chirality as well. Due to the absence of tadpole constraints for four-form fluxes on G_2 -manifolds, the local scenario for fluxes in type II string theories proposed in ref. [86] is readily realized on the level of the non-compact asymptotically cylindrical Calabi–Yau summands $X_{L/R}$. We expect that non-trivial background four-form fluxes provides for a much more intricate and genuine $\mathcal{N} = 1$ gauge theory branch structure, similarly as in refs. [76, 87]. All these effects come with different — partially exponentially suppressed — scales, which exhibit potentially attractive hierarchies. A systematic analysis of phenomenologically attractive $\mathcal{N} = 1$ or even $\mathcal{N} = 0$ interaction terms in the context of M-theory on twisted connected sum G_2 -manifolds is beyond the scope of this paper and will be addressed elsewhere.

While in type II Calabi–Yau threefold compactifications we arrive at four-dimensional $\mathcal{N} = 2$ effective supergravity theories with two massless gravitinos realizing extended supersymmetry, breaking the $\mathcal{N} = 2$ gravity multiplet down to the $\mathcal{N} = 1$ is rather non-trivial, see for instance the discussion in ref. [88]. In our case, however, the obtained four-dimensional supergravity theory has already minimal supersymmetry. It is only the gauge theory sectors in the Kovalev limit that approximately exhibit extended global supersymmetries. Therefore, introducing background fluxes to break supersymmetry in the gauge theory sectors is much simpler as in the type II Calabi–Yau threefold compactifications. In particular turning on background fluxes

³⁵A similar analysis of moduli dependent matching conditions is required for building blocks (Z, S) arising from general weak Fano threefold for which Beauville’s theorem [57] is not applicable.

resembles to a great extent the type II scenario of ref. [86], in which, however, gravity is decoupled.

Let us point out a further potentially phenomenologically attractive possibility. As explained in Section 4.1 the separation of the two sectors X_L and X_R in Figure 3.1 is controlled by the real part of the Kovalevton κ . Together with the local construction of the spectra on $X_{L/R}$ described in Section 6, this offers the possibility to consider a hidden and a visible sector and to employ the mechanism on mediation of supersymmetry breaking only in the gravitational sector with a controllable scale set by Kovalevton κ . Or alternatively, as there is an anomaly inflow mechanism in the local theories [32, 33], one could use the anomaly mediation of supersymmetry breaking as proposed in [89].

Finally we comment on the possible relation of the twisted connected sum construction to other non-perturbative descriptions of $\mathcal{N} = 1$ theories. In lower dimensions the algebraic-geometrical approach towards the Hořava–Witten setup describes a duality to F-theory [90]. Namely, certain Calabi–Yau compactifications of the heterotic string are dual to F-theory on elliptically fibered Calabi–Yau fourfolds in a particular stable degeneration limit [91]. To obtain four-dimensional $\mathcal{N} = 1$ supergravity theory this heterotic–F-theory correspondence is realized on the level of elliptically-fibered Calabi–Yau fourfolds. It is intriguing to observe that such Calabi–Yau fourfolds in the stable degeneration limit are obtained by gluing a pair of suitably chosen Fano fourfolds along their mutual anti-canonical Calabi–Yau threefold divisor [92, 93]. This construction of Calabi–Yau fourfolds in the stable degeneration limit shows a certain resemblance — yet in one real dimension higher — to twisted connected sum G_2 -manifolds in the Kovalev limit. It would be interesting to see, if such a speculation could be made precise, namely establishing a duality between M-theory on G_2 -manifolds in the Kovalev limit and F-theory on elliptically-fibered Calabi–Yau fourfolds in a certain degeneration limit.

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A Kaluza-Klein reduction of fermionic terms

In this appendix we present the Kaluza–Klein reduction of the fermionic terms obtained from the compactification of the eleven-dimensional supergravity action (2.22)

on a seven-dimensional G_2 -manifold. We analyze the four-dimensional fermionic spectrum, and explicitly derive the zero modes of the eleven-dimensional Rarita–Schwinger field compactified on the G_2 -manifold. We further determine the holomorphic superpotential induced by internal four-form fluxes from certain fermionic interaction terms. Compared to the purely bosonic interactions with quadratic dependences on the superpotential, the fermionic interactions are linear in the superpotential, see for instance ref. [51]. Thus, analyzing the fermionic interactions as opposed to the purely bosonic ones is more tractable for determining the superpotential.

A.1 Definitions and useful relations

Following the definition and conventions of ref. [20], we first spell out the properties of the representation of the used eleven-, seven- and four-dimensional gamma matrices. The eleven-dimensional gamma matrices are represented by 32-dimensional matrices, which satisfy the usual Clifford algebra

$$\{\hat{\Gamma}_M, \hat{\Gamma}_N\} = 2g_{MN} , \quad (\text{A.1})$$

with the eleven-dimensional Lorentzian metric g_{MN} . In the chosen 32-dimensional Majorana representation the gamma matrices further obey [20]

$$\hat{\Gamma}_0 \cdots \hat{\Gamma}_{10} = \mathbb{I} , \quad (\text{A.2})$$

in terms of the 32-dimensional identity matrix \mathbb{I} . With the compactification ansatz $M^{1,10} = \mathbb{M}^{1,3} \times Y$ the eleven-dimensional gamma matrices split into two sets of commuting gamma matrices, i.e.,

$$\hat{\Gamma}_M = (\hat{\Gamma}_\mu, \hat{\Gamma}_m) , \quad \hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{I} , \quad \hat{\Gamma}_m = \gamma \otimes \gamma_m , \quad (\text{A.3})$$

where \mathbb{I} is the seven-dimensional identity matrix, γ_μ , $\mu = 0, 1, 2, 3$, are the four-dimensional imaginary gamma matrices, γ_m , $m = 4, \dots, 10$, are purely imaginary seven-dimensional gamma matrices satisfying $\gamma_4 \cdots \gamma_{10} = i$. Furthermore, we define $\gamma = (i/4!) \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$ as the four-dimensional chirality matrix, which is purely imaginary and satisfies $\gamma^2 = 1$.

The four- and seven-dimensional gamma matrices satisfy the Clifford algebra in their corresponding dimensions

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} , \quad \{\gamma_m, \gamma_n\} = 2g_{mn} . \quad (\text{A.4})$$

Here we use the Minkowski metric $\eta_{\mu\nu}$ with signature $(-1, +1, +1, +1)$ and g_{mn} denotes the Riemannian metric of the seven-dimensional compactification space.

We define the anti-symmetrized product of eleven-dimensional gamma matrices as

$$\hat{\Gamma}^{M_1 \cdots M_n} = \hat{\Gamma}^{[M_1} \cdots \hat{\Gamma}^{M_n]} , \quad (\text{A.5})$$

and we use the same notation for the anti-symmetrized products of four- and seven-dimensional gamma matrices, i.e., $\gamma^{\mu_1 \dots \mu_n} = \gamma^{[\mu_1 \dots \mu_n]}$ and $\gamma^{m_1 \dots m_n} = \gamma^{[m_1 \dots m_n]}$. For the decomposition of $\hat{\Gamma}^{MNP}$ into lower-dimensional gamma matrices we arrive at the useful relation

$$\begin{aligned} \hat{\Gamma}^{MNP} = & (\gamma^{\mu\nu\rho} \otimes \mathbb{I}) + (\gamma^{\mu\nu-} \otimes \gamma^p) + (\gamma^{\mu-\rho} \otimes \gamma^n) + (\gamma^{-\nu\rho} \otimes \gamma^m) \\ & + \frac{1}{3}(\gamma^\rho \otimes \gamma^{mn} + \gamma^\nu \otimes \gamma^{pm} + \gamma^\mu \otimes \gamma^{np}) \\ & + \gamma \otimes \gamma^{mnp} , \end{aligned} \quad (\text{A.6})$$

where the index ‘ $-$ ’ refers to the four-dimensional chirality matrix γ .

For the forthcoming zero mode analysis we record here a few useful identities. First of all, we record a few useful identities among products of anti-symmetrized gamma matrices, namely

$$\begin{aligned} \gamma^{mnp} \gamma^q &= \gamma^{mnpq} + 3g^{q[m} \gamma^{np]} , \\ \gamma^{mnp} \gamma^{qr} &= \gamma^{mnpqr} - 3(g^{q[m} \gamma^{np]r} - g^{r[m} \gamma^{np]q}) + 6g^{q[m} \gamma^n g^{p]r} . \end{aligned} \quad (\text{A.7})$$

Furthermore, the G_2 -structure φ fulfills the contraction relations

$$\varphi_{nmp} \varphi^{mpq} = 6 \delta_n^q , \quad \varphi_{nmp} \varphi^{pqr} = \Phi_{mn}{}^{qr} - \delta_n^p \delta_m^q + \delta_n^q \delta_m^p , \quad (\text{A.8})$$

with the Hodge dual form $\Phi = *\varphi$, and the Fierz identity

$$\gamma^{mn} \eta = -i \varphi^{mnp} \gamma_p \eta , \quad (\text{A.9})$$

in terms of the covariantly constant spinor η . Finally, the Levi-Civita connection ∇ , the exterior derivative d and its adjoint d^\dagger fulfill the relations

$$\begin{aligned} (dA)_{n_1 \dots n_{p+1}} &= (p+1) \nabla_{[n_1} A_{n_2 \dots n_{p+1}]} , \\ (d^\dagger A)_{n_1 \dots n_{p-1}} &= -\nabla^m A_{mn_1 \dots n_{p-1}} , \end{aligned} \quad (\text{A.10})$$

for any the p -form $A = \frac{1}{p!} A_{n_1 \dots n_p} dy^{n_1} \wedge \dots \wedge dy^{n_p}$.

A.2 G_2 representations and the G_2 Rarita–Schwinger bundle

In this section we present further details for the decomposition of the global section (2.18) of the Rarita–Schwinger bundle $T^*Y \otimes SY$ on G_2 -manifolds, as introduced in Section 2.

First of all, the differential p -forms on a manifold with G -structure fall into irreducible representations with respect to the structure group G . Specifically, for a seven-manifold with G_2 -structure the spaces of differential p -forms Λ^p decompose according to ref. [7], namely

$$\begin{aligned} \Lambda^0 &= \Lambda_1^0 , \quad \Lambda^1 = \Lambda_7^1 , \quad \Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2 , \quad \Lambda^3 = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3 , \\ \Lambda^4 &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4 , \quad \Lambda^5 = \Lambda_7^5 \oplus \Lambda_{14}^5 , \quad \Lambda^6 = \Lambda_7^6 , \quad \Lambda^7 = \Lambda_1^7 , \end{aligned} \quad (\text{A.11})$$

where the summands $\Lambda_{\mathbf{q}}^p$ are p -forms transforming in the q -dimensional irreducible representations of the structure group G_2 . As indicated in the arrangement of the form space $\Lambda_{\mathbf{q}}^p$ in eq. (A.11), the Hodge star $*$ provides for an isometry between $\Lambda_{\mathbf{q}}^p$ and $\Lambda_{\mathbf{q}}^{7-p}$. The differential p -form spaces $\Lambda_{\mathbf{7}}^p$ are isomorphic to each other for $p = 1, \dots, 6$. Moreover, $\Lambda_{\mathbf{1}}^3$ and $\Lambda_{\mathbf{1}}^4$ are generated by φ and $*\varphi$, respectively. For a compact G_2 -manifold Y equipped with a torsion-free G_2 -structure the de Rham cohomologies $H^p(Y, \mathbb{R})$ have a similar decomposition into $H_{\mathbf{q}}^p(Y, \mathbb{R})$ with harmonic representatives, see, e.g., ref. [7],

$$\begin{aligned} H^2(Y, \mathbb{R}) &= H_{\mathbf{7}}^2(Y, \mathbb{R}) \oplus H_{\mathbf{14}}^2(Y, \mathbb{R}) , \\ H^3(Y, \mathbb{R}) &= H_{\mathbf{1}}^3(Y, \mathbb{R}) \oplus H_{\mathbf{7}}^3(Y, \mathbb{R}) \oplus H_{\mathbf{27}}^3(Y, \mathbb{R}) , \\ H^4(Y, \mathbb{R}) &= H_{\mathbf{1}}^4(Y, \mathbb{R}) \oplus H_{\mathbf{7}}^4(Y, \mathbb{R}) \oplus H_{\mathbf{27}}^4(Y, \mathbb{R}) , \\ H^5(Y, \mathbb{R}) &= H_{\mathbf{7}}^5(Y, \mathbb{R}) \oplus H_{\mathbf{14}}^5(Y, \mathbb{R}) . \end{aligned} \tag{A.12}$$

Notice that $H_{\mathbf{1}}^3(Y, \mathbb{R}) = \langle [\varphi] \rangle$ and $H_{\mathbf{1}}^4(Y, \mathbb{R}) = \langle [* \varphi] \rangle$. Moreover, $H_{\mathbf{q}}^p(Y, \mathbb{R}) \cong H_{\mathbf{q}}^{7-p}(Y, \mathbb{R})$, which implies for the Betti numbers $b_{\mathbf{q}}^p(Y) = b_{\mathbf{q}}^{7-p}(Y)$ and $b_{\mathbf{1}}^3(Y) = b_{\mathbf{1}}^4(Y) = 1$. If the holonomy group is G_2 and not a subgroup thereof, we further have $H_{\mathbf{7}}^p = \{0\}$ for $p = 1, \dots, 6$.

Let us now turn to the Rarita–Schwinger bundle on G_2 -manifolds, which — due to the covariantly constant spinor η — becomes reducible, namely $T^*Y \otimes SY \cong T^*Y \otimes (T^*Y \oplus \mathbb{R})$. This allows us to make the following identification

$$\begin{aligned} T^*Y \otimes SY &\cong T^*Y \otimes (T^*Y \oplus \mathbb{R}) \\ &= (T^*Y \otimes T^*Y) \oplus T^*Y \\ &= \text{Sym}^2(T^*Y) \oplus \Lambda^2 T^*Y \oplus T^*Y , \end{aligned} \tag{A.13}$$

where $\text{Sym}^2(T^*Y)$ is the symmetric two-tensor on Y and $\Lambda^2 T^*Y$ is the space of two-forms. Furthermore, it is shown in ref. [94] that $\text{Sym}^2(T^*Y) \cong \Lambda_{\mathbf{1}}^3 \oplus \Lambda_{\mathbf{27}}^3$. Since the spaces $\Lambda_{\mathbf{7}}^2$ and $\Lambda_{\mathbf{7}}^3$ are isomorphic to the cotangent bundle $\Lambda_{\mathbf{7}}^1$, we arrive at

$$T^*Y \otimes SY \cong \Lambda_{\mathbf{1}}^3 \oplus \Lambda_{\mathbf{27}}^3 \oplus \Lambda_{\mathbf{14}}^2 \oplus \Lambda_{\mathbf{7}}^1 . \tag{A.14}$$

This decomposition of the Rarita–Schwinger bundle justifies the ansatz for the global Rarita–Schwinger section (2.18) in Section 2.

A.3 The massless four-dimensional fermionic spectrum

We are now ready to determine the four-dimensional fermionic terms in the dimensional reduction of eleven-dimensional supergravity action on G_2 -manifolds. We focus on the four-dimensional fermionic kinetic and mass terms.

Let us perform the dimensional reduction of the Rarita–Schwinger kinetic term for the gravitino $\hat{\Psi}$, which is given by the third term of the first line in (2.22). Inserting

the expansion (2.16) for the gravitino $\hat{\Psi}$ and the relation (A.6) we obtain

$$\begin{aligned}
-\frac{1}{2\kappa_{11}^2} \int *_{11} i \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNP} \hat{\nabla}_N \hat{\Psi}_P &= -\frac{i}{2\kappa_{11}^2} \int_{\mathbb{M}^{1,3}} *_{4} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho^* \int_Y *_{7} \bar{\zeta} \zeta \\
&\quad - \frac{i}{2\kappa_{11}^2} \int_{\mathbb{M}^{1,3}} *_{4} \bar{\psi}_\mu \gamma^{\mu-\rho} \psi_\rho^* \int_Y *_{7} \bar{\zeta} \gamma^n \nabla_n \zeta \\
&\quad - \frac{i}{2\kappa_{11}^2} \int_{\mathbb{M}^{1,3}} *_{4} \frac{1}{3} \bar{\chi} \gamma^\nu \nabla_\nu \chi^* \int_Y *_{7} \bar{\zeta}_m^{(1)} \gamma^{pm} \zeta_p^{(1)} \quad (\text{A.15}) \\
&\quad - \frac{i}{2\kappa_{11}^2} \int_{\mathbb{M}^{1,3}} *_{4} \bar{\chi} \gamma \chi^* \int_Y *_{7} \bar{\zeta}_m^{(1)} \gamma^{mnp} \nabla_n \zeta_p^{(1)} \\
&\quad + \text{c.c.} .
\end{aligned}$$

The resulting terms comprise the kinetic and mass terms for both the four-dimensional gravitinos ψ_μ — the first and second line on the right hand side of eq. (A.15), respectively — and the four-dimensional fermions χ — the third and fourth line on the right hand side of eq. (A.15), respectively. It also gives rise to mixed terms between ψ and χ . However, since such mixed terms are not present in standard four-dimensional supergravity theories, they have been neglected in our analysis.³⁶

Now, we turn to the discussion of the massless four-dimensional fermionic spectrum, which is obtained from the zero modes of the Dirac operator $\not{D} = \gamma^n \nabla_n$ and the Rarita–Schwinger operator $\not{D}^{\text{RS}} = \gamma^{mnp} \nabla_n$, i.e., $\not{D}\zeta = 0$ and $\not{D}^{\text{RS}}\zeta^{(1)} = 0$.

With the ansatz (2.17) for the section ζ of the spin bundle SY , we arrive at the zero modes equation

$$\not{D}\zeta = (\nabla_n a_m) \gamma^n \gamma^m \eta + (\partial_n b) \gamma^n \eta = \nabla_{[n} a_{m]} \gamma^{nm} \eta + (\nabla^n a_n) \eta + (\partial_n b) \gamma^n \eta = 0 , \quad (\text{A.16})$$

which — due to the linear independence of η , $\gamma^n \eta$, and $\gamma^{nm} \eta$ — yields for the coefficient one-form $a(y) = a_n(y) dy^n$ and the function $b(y)$ together with eqs. (A.10)

$$da(y) = 0 , \quad d^\dagger a(y) = 0 , \quad db(y) = 0 . \quad (\text{A.17})$$

The first two equations imply that $a(y)$ must be a harmonic one-form, whereas the last equation determines the function $b(y)$ to be constant. As there are no harmonic one forms on the G_2 -manifold Y , the covariantly constant spinor η furnishes the only zero mode in the spin bundle SY . This zero mode gives rise to the massless four-dimensional gravitino field ψ_μ and its conjugate ψ_μ^* of the four-dimensional massless $\mathcal{N} = 1$ gravity multiplet listed in Table 2.1.

³⁶Actually, one should perform a redefinition of Ψ_μ with $\Psi_\mu \rightarrow \Psi'_\mu = \Psi_\mu + \hat{\Gamma}_\mu \hat{\Gamma}^m \Psi_m$ in order for such terms to cancel out. However, we do not consider this field redefinition as such a shift does not affect the gravitino mass [20, 95].

Analogously, by acting with the Rarita–Schwinger operator \mathcal{D}^{RS} on the ansatz (2.18) for $\zeta^{(1)}$ and using eqs. (A.7), (A.8) and (A.9), we arrive after a straightforward but somewhat tedious calculation at

$$\begin{aligned}
\mathcal{D}^{\text{RS}} \zeta^{(1)} = & (\nabla_{[n} b_{m]}^7) \gamma^{mnp} dy_p \otimes \eta - (\nabla^n a_{np}^{14}) dy_m \otimes \gamma^{pm} \eta \\
& + (\nabla_{[n} a_{pq]}^{14}) \left(\gamma^{mnpq} dy_m \otimes \eta + \frac{1}{2} dy^q \otimes \gamma^{np} \eta \right) \\
& - \frac{3i}{2} (\nabla^n a_{npq}^{28}) (dy_m \otimes \gamma^{pqm} \eta - dy^p \otimes \gamma^q \eta) \\
& + (\nabla_{[m} a_{npq]}^{28}) \left(\frac{3i}{4} \gamma^{mnpqr} dy_r \otimes \eta + 2i dy^m \otimes \gamma^{npq} \eta \right) \\
& + \partial_p (\text{tr}_g a_{(mn)}^{28}) \left(\frac{439}{72} dy_q \otimes \gamma^{pq} \eta - 6i dy^p \otimes \eta \right) ,
\end{aligned} \tag{A.18}$$

in terms of the singlet $\text{tr}_g a_{(mn)}^{28} = a_{nm}^{28} g^{nm}$ and the three-form $a_{[nmp]}$

$$a_{[mnp]}^{28} = g^{rs} a_{r[m}^{28} \varphi_{np]s} , \quad a_{(nm)}^{28} = \frac{3}{4} a_{[npq]}^{28} \varphi^{pqr} g_{rm} - \frac{439}{84} g_{nm} a_{[pqr]}^{28} \varphi^{pqr} . \tag{A.19}$$

Let us now analyze the zero modes of the Rarita–Schwinger section $\zeta^{(1)}$ from eq. (A.18). The one-form $b(y) = b_n(y) dy^n$ does not contribute any zero modes, because with eq. (A.10) such a zero mode must be a closed one-form $db(y) = 0$. Furthermore, due to $b_1(Y) = 0$ it also must be exact $b(y) = df(y)$. However, an exact one-form $df(y)$ furnishes no physical degrees of freedom as it can be removed by a gauge transformation of the Rarita–Schwinger section, i.e., $\zeta^{(1)} \rightarrow \zeta^{(1)} - \nabla(f(y) \otimes \eta)$. For the remaining tensors we find with the help of eqs. (A.10) that the zero modes of the Rarita–Schwinger operator \mathcal{D}^{RS} are given by

$$\begin{aligned}
da^{14}(y) &= 0 , & d^\dagger a^{14}(y) &= 0 , \\
da^{28}(y) &= 0 , & d^\dagger a^{28}(y) &= 0 ,
\end{aligned} \tag{A.20}$$

in terms of the two-form $a^{14}(y) = \frac{1}{2} a_{nm}^{14}(y) dy^n \wedge dy^m$ and the three-form $a^{28}(y) = \frac{1}{6} a_{nmp}^{28}(y) dy^n \wedge dy^m \wedge dy^p$. Thus, the zero modes are in one-to-one correspondence with harmonic two-forms $a^{14}(y)$ and three-forms $a^{28}(y)$ on the G_2 -manifolds Y , where the harmonic property of $a^{28}(y)$ implies that the symmetric tensor $a_{(nm)}^{28}$ must be solutions to the Lichnerowicz Laplacian as well, c.f., eq. (2.10). Altogether, we can therefore deduce from the cohomology of the G_2 -manifold Y the fermionic zero modes listed in Table 2.1.

A.4 The flux-induced holomorphic superpotential

Let us now determine the holomorphic superpotential generated by a cohomologically non-trivial four-form background flux G on the G_2 -manifold Y , which is locally given

by $d\hat{C}$. The superpotential can be read off from the four-dimensional gravitino mass term (2.35). Such a term arises from the dimensional reduction of the fourth term in the eleven-dimensional action (2.22). That is to say we find

$$\begin{aligned} -\frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS} \hat{\Psi}_N (d\hat{C})_{[PQRS]} &\supset -\frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\hat{\Psi}}_\mu \hat{\Gamma}^{\mu\nu pqrs} \hat{\Psi}_\nu (d\hat{C})_{[pqrs]} \\ &= -\frac{1}{192\kappa_{11}^2} \int *_{11} (\bar{\psi}_\mu + \bar{\psi}_\mu^*) \bar{\zeta} \gamma^{\mu\nu} \gamma^{pqrs} (\psi_\nu + \psi_\nu^*) \zeta (d\hat{C})_{[pqrs]} . \end{aligned} \quad (\text{A.21})$$

Since there are no harmonic one-forms on the G_2 -manifold Y , we can identify the spinorial section ζ with the unique covariant constant spinor η on the G_2 -manifold Y , c.f., Section A.3. Furthermore, we notice that the covariantly constant three-form φ and its Hodge dual four-form $\Phi = *\varphi$ are bilinear in η , namely $\varphi_{[mnp]} = i\bar{\eta}\gamma_{mnp}\eta$ and $\Phi_{[mnpq]} = (*\varphi)_{[mnpq]} = -\bar{\eta}\gamma_{mnpq}\eta$ such that

$$\begin{aligned} -\frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS} \hat{\Psi}_N (d\hat{C})_{[PQRS]} \\ \supset \frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu^* \Phi_{[pqrs]} (d\hat{C})^{[pqrs]} + \text{c.c.} \end{aligned} \quad (\text{A.22})$$

To arrive at the four-dimensional $\mathcal{N} = 1$ supergravity action in the conventional Einstein frame, we employ the Weyl rescalings

$$g_{\mu\nu} \rightarrow \frac{g_{\mu\nu}}{\lambda_0(S^i)} , \quad \gamma^\mu \rightarrow \sqrt{\lambda_0(S^i)} \gamma^\mu , \quad \psi_\mu \rightarrow \frac{\psi_\mu}{(\lambda_0(S^i))^{1/4}} . \quad (\text{A.23})$$

Using the dimensionless volume factor defined in eq. (2.25) and $\kappa_{11}^2 = V_{Y_0} \kappa_4^2$ in terms of the reference volume $V_{Y_0} = V_Y(S_0^i)$ defined in Section 2, we obtain

$$\begin{aligned} -\frac{1}{192\kappa_{11}^2} \int *_{11} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS} \hat{\Psi}_N (d\hat{C})_{[PQRS]} \\ \supset \frac{1}{8\lambda_0^{3/2} V_{Y_0} \kappa_4^2} \int_Y G \wedge \varphi \int_{\mathbb{M}^{1,3}} *_4 \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu^* + \text{c.c.} \end{aligned} \quad (\text{A.24})$$

Therefore, with the Kähler potential $K = -3\ln\lambda_0$ derived in Section 2 and the gravitino mass term (2.35), we deduce the following superpotential contribution

$$W \supset \frac{1}{4V_{Y_0}} \int_Y G \wedge \varphi . \quad (\text{A.25})$$

Note that the derived term is not holomorphic in the four-dimensional chiral coordinates, as it couples to the three-form φ and not its complexification. To arrive at the full superpotential we must render the moduli dependence holomorphic in terms of the replacement $\varphi \rightarrow \varphi + iC$, which is in accord with the deduced chiral coordinates (2.31). This proposed replacement is in agreement with the domain wall tensions interpolating between distinct flux vacua [8, 96]. Thus, altogether we arrive at the flux-induced superpotential (2.36).

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